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CONTENTS

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ROY DUBISCH, *Editor*

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EDITORIAL COMMENTS

1. In the next issue of the MATHEMATICS MAGAZINE a section of book reviews will be reinstated under the editorship of Professor Dmitri Thoro of San Jose State College. In conformity with recommendations by the Committee on Publications and the Board of Governors, reviews will be confined to books of interest to students and teachers of the first two years of college mathematics. This will include not only calculus and pre-calculus textbooks but also other books at this general level. Some overlap with the books reviewed in the *Monthly* and the *Mathematics Teacher* is to be expected and, indeed, may be desirable, but it is hoped that the overlap will be minimal.

2. It will be of great assistance to the editors if authors will observe the following points in the preparation of their manuscripts:

The MATHEMATICS MAGAZINE welcomes papers presenting valid mathematics, of rather general interest, at a level intelligible to persons with an undergraduate major in mathematics. Some novelty of content, viewpoint, or arrangement is essential. Expository articles are particularly desired. State the context and the principal aim of the paper early. Address yourself quite explicitly to the reader described above, communicating your ideas to him clearly and attractively.

The title should be brief and meaningful. Since the title will be quoted and reproduced by laymen, it should contain no symbols unfamiliar to laymen.

Articles should be typewritten, double-spaced, on $8\frac{1}{2}\times 11''$ paper of very good quality. Submit the original (and a duplicate if convenient) keeping a complete copy for yourself. To avoid loss and delay notify us of any change of address.

The typescript should be prepared with extreme care. Misprints are highly obnoxious; so are dangling participles. Put name and address between title and text. Put references at the end with bracketed citations in the text. Avoid footnotes; instead, use clearly designated remarks in the text. Put acknowledgments at the very end, just before the bibliography. Be generous with spacing and displays. Keep notation simple. For a matrix the notation $[a_{jk}]$ is recommended, with $\det[a_{jk}]$ or $|a_{jk}|$ for the determinant. On doubtful questions regarding format or notation, observe practices in current issues of the MATHEMATICS MAGAZINE, or consult the applicable sections of the "Manual for Authors of Mathematical Papers" printed in the September, 1962 issue of the *Bulletin of the American Mathematical Society* (pp. 429–444).

A METHOD OF ESTABLISHING CERTAIN IRRATIONALITIES

E. A. MAIER AND IVAN NIVEN, University of Oregon

1. Introduction. H. Steinhaus [1] has given a proof of the irrationality of $\sqrt{2}$ that approaches the question through inequalities rather than divisibility. A variation of this proof is given in section 2; the form given here is more suitable to extension to other questions. Several of these extensions are given in sections 3, 4, and 5. Actually, Theorem 2 of section 5 includes all previous results as special cases. However the chain of results leading to Theorem 2 reveals how we arrived at the proofs. The only background results needed for sections 2 and 3 are a few basic propositions about inequalities; beyond section 3 the division algorithm is also needed.

2. The irrationality of $\sqrt{2}$. Suppose that $\sqrt{2}$ is rational, say

$$\sqrt{2} = a/b, \tag{1}$$

where a and b are positive integers. Suppose that b is the smallest positive integer for which there is a representation (1). From (1) and the fact that $\sqrt{2}$ lies between 1 and 2 we see that

$$1 < \frac{a}{b} < 2, \qquad b < a < 2b, \qquad 0 < a - b < b. \tag{2}$$

Then we note from (1) that

$$a^2 = 2b^2,\ a^2 - ab = 2b^2 - ab,\ a(a - b) = b(2b - a),\ \sqrt{2} = \frac{a}{b} = \frac{2b - a}{a - b}\,. \tag{3}$$

Thus $\sqrt{2}$ has been expressed in another rational form with positive denominator $a-b$, smaller than b, contrary to the minimal property assumed for b. Thus $\sqrt{2}$ is irrational.

3. Extension to $\sqrt{d}$. This proof generalizes at once from $\sqrt{2}$ to $\sqrt{d}$, where d is a positive integer not a perfect square. Thus $\sqrt{d}$ lies between two consecutive integers, say

$$t < \sqrt{d} < t + 1.$$

Then if (1) is replaced by $\sqrt{d}=a/b$, where the positive integer b is minimal, we see that (2) can be replaced by

$$t < \frac{a}{b} < t + 1, \qquad tb < a < tb + b, \qquad 0 < a - tb < b.$$

Hence the equations leading up to (3) are replaced by

$$a^2 = db^2, \quad a^2 - tab = db^2 - tab, \quad a(a - tb) = b(db - ta), \quad \sqrt{d} = \frac{a}{b} = \frac{db - ta}{a - tb}$$

and the contradiction is analogous to that in the $\sqrt{2}$ case.

4. The extension to nth roots. The minimum property of the integer b in the foregoing proofs amounts to this: Let α be a rational number, and suppose that $\alpha=a/b$ is the representation of α having minimum positive integer b in the denominator. Then b is seen to be the smallest positive integer such that $b\alpha$ is an integer. In what follows it will be convenient to use this interpretation of b.

To generalize the proof to the case of nth roots, we make use of the division algorithm: Given any integers a and b, with b positive, there exist integers q and r such that $a=bq+r$, $0\leqq r<b$.

LEMMA 1. *Let α be rational, and let b be the least positive integer such that $b\alpha$ is an integer, say $b\alpha=a$. For any nonnegative integer m, if s is the least positive integer such that $sa^m\alpha$ is an integer, then $s=b$.*

Proof. The proof is by induction on m. Note that the result holds for $m=0$. We assume that b is the least positive integer such that $ba^m\alpha$ is an integer, and that s is the least positive integer such that $sa^{m+1}\alpha$ is an integer, and we prove that $s=b$. Note that $s\leqq b$ since $ba^{m+1}\alpha$ is an integer. Apply the division algorithm to sa^{m+1} and b to get integers q and r satisfying

$$sa^{m+1} = bq + r, \qquad 0 \leqq r < b.$$

Thus we see that

$$r\alpha = (sa^{m+1} - bq)\alpha = sa^{m+1}\alpha - aq. \tag{4}$$

Thus $r\alpha$ is an integer, and so $r=0$ by the minimal property of b. Hence (4) reduces to

$$0 = sa^{m+1}\alpha - aq, \qquad sa^m\alpha = q.$$

Thus $s\geqq b$ by the induction hypothesis, and hence $s=b$.

THEOREM 1. *Let n and d be positive integers greater than 1 such that $\sqrt[n]{d}$ is not an integer. Then $\sqrt[n]{d}$ is irrational.*

Proof. Let $\sqrt[n]{d}$ lie between the consecutive integers t and $t+1$, so that

$$t < \sqrt[n]{d} < t + 1. \tag{5}$$

Assume that $\sqrt[n]{d}$ is rational, and let b be the least positive integer such that $b\sqrt[n]{d}$ is an integer, say $b\sqrt[n]{d}=a$; then (5) implies that

$$t < \frac{a}{b} < t + 1, \qquad tb < a < tb + b, \qquad 0 < a - tb < b. \tag{6}$$

Furthermore we observe that (writing α for $\sqrt[n]{d}$)

$$(a - tb)a^{n-2}\alpha = a^{n-1}\alpha - ta^{n-2}b\alpha = b^{n-1}d - ta^{n-1},$$

an integer. But in view of the last part of (6) this contradicts Lemma 1, where m is here $n-2$.

5. Algebraic integers. A number α is said to be an algebraic integer if it satisfies some equation of the type

$$\alpha^n + c_{n-1}\alpha^{n-1} + c_{n-2}\alpha^{n-2} + \cdots + c_1\alpha + c_0 = 0, \tag{7}$$

with integer coefficients. For example the number $\sqrt[n]{d}$ of the preceding section is an algebraic integer. We confine attention to those algebraic numbers that are real numbers.

THEOREM 2. *Let α be a real algebraic integer. If α is not a rational integer* (*i.e., one that satisfies an equation of type* (7) *of degree $n=1$*), *then α is irrational.*

Proof. Suppose that α satisfies (7), and that α is rational. Let b be the least positive integer such that $b\alpha$ is an integer, say $b\alpha=a$. Then, as in the proof of Theorem 1, there exists an integer t such that

$$0 < a - tb < b. \tag{8}$$

Also by (7) we note that $b^{n-1}\alpha^n$ is an integer:

$$\begin{aligned} b^{n-1}\alpha^n &= b^{n-1}[-c_{n-1}\alpha^{n-1} - c_{n-2}\alpha^{n-2} - \cdots - c_1\alpha - c_0] \\ &= -c_{n-1}(b\alpha)^{n-1} - c_{n-2}b(b\alpha)^{n-2} - \cdots - c_1b^{n-2}(b\alpha) - c_0b^{n-1}. \end{aligned}$$

But then

$$(a-tb)a^{n-2}\alpha = a^{n-1}\alpha - ta^{n-2}b\alpha = b^{n-1}\alpha^n - ta^{n-1}$$

is an integer which, in view of (8), contradicts Lemma 1.

Reference

1. H. Steinhaus, Mathematical snapshots, Stechert-Hafner, New York, 1938, p. 132.

NOTE ON SQUARES AND CUBES

J. ALLARD, University of Sherbrooke

Geometrical squares and cubes are members of the N-dimensional family expressed by

$$\sum_{i=1}^{N}\left(\frac{x_i}{a_i}\right)^{2n_i} = C. \tag{1}$$

In the above equation, a_i and C are constants with n_i a parameter. Several well-known geometrical figures can be obtained as particular and limiting cases of equation (1). When $N=2$, that is for two-dimensional problems, all members of a particular family of pseudo-squares have four points of tangency at the axis. A geometrical square can be represented in cartesian coordinates by the expression

$$\lim_{n\to\infty} \sum_{i=1}^{2}\left(\frac{x_i}{r}\right)^{2n} = 1. \tag{2}$$

5. Algebraic integers. A number α is said to be an algebraic integer if it satisfies some equation of the type

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with integer coefficients. For example the number $\sqrt[n]{d}$ of the preceding section is an algebraic integer. We confine attention to those algebraic numbers that are real numbers.

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Proof. Suppose that α satisfies (7), and that α is rational. Let b be the least positive integer such that $b\alpha$ is an integer, say $b\alpha = a$. Then, as in the proof of Theorem 1, there exists an integer t such that

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$$\begin{aligned} b^{n-1}\alpha^n &= b^{n-1}[-c_{n-1}\alpha^{n-1} - c_{n-2}\alpha^{n-2} - \cdots - c_1\alpha - c_0] \\ &= -c_{n-1}(b\alpha)^{n-1} - c_{n-2}b(b\alpha)^{n-2} - \cdots - c_1 b^{n-2}(b\alpha) - c_0 b^{n-1}. \end{aligned}$$

But then

$$(a - tb)a^{n-2}\alpha = a^{n-1}\alpha - ta^{n-2}b\alpha = b^{n-1}\alpha^n - ta^{n-1}$$

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$$\lim_{n\to\infty} \sum_{i=1}^{2}\left(\frac{x_i}{r}\right)^{2n} = 1. \tag{2}$$

A pseudo-square, that is, a curve which approaches a geometrical square, can be plotted from the equation

$$\sum_{i=1}^{2} x_i^{2n} = r^{2n}, \tag{3}$$

where r is the distance from the center to the x or y axis intercept and n is a positive integer greater than 2. Figures Ia and Ib illustrate the geometrical loci obtained with different n and show that the only difference between the circle and the square lies in the radius of curvature which is constant for the circle and varies for all pseudo-squares.

The area under these two-dimensional curves can be obtained by the following simple method. In

$$A = \int_0^r (r^{2n} - x^{2n})^{1/2n} dx$$

let $z = (x/r)^{2n}$ and substitute to get

$$A = \frac{4r^2}{2n} \int_0^1 (1 - z)^{1/2n} z^{1/2n-1} dz$$

which can be expressed in terms of the gamma function as

$$A = \frac{4r^2 \{\Gamma(1/2n + 1)\}^2}{\Gamma(1/n + 1)}. \tag{4}$$

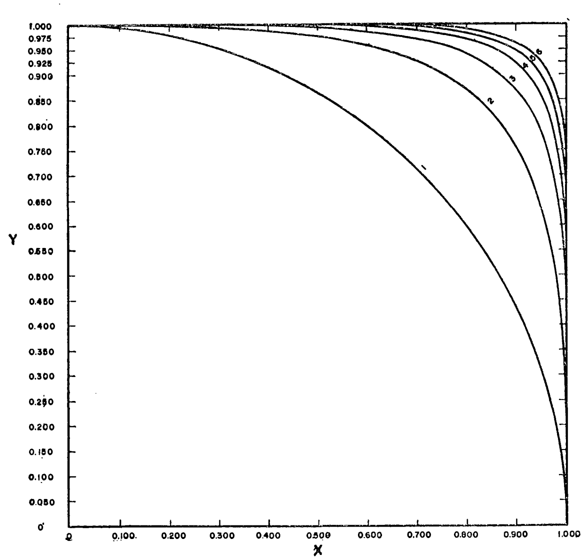

FIG. Ia. First quadrant plotting of $x^{2n}+y^{2n}=1$ for $n=1, 2, 3, 4, 5, 6$ and ∞.

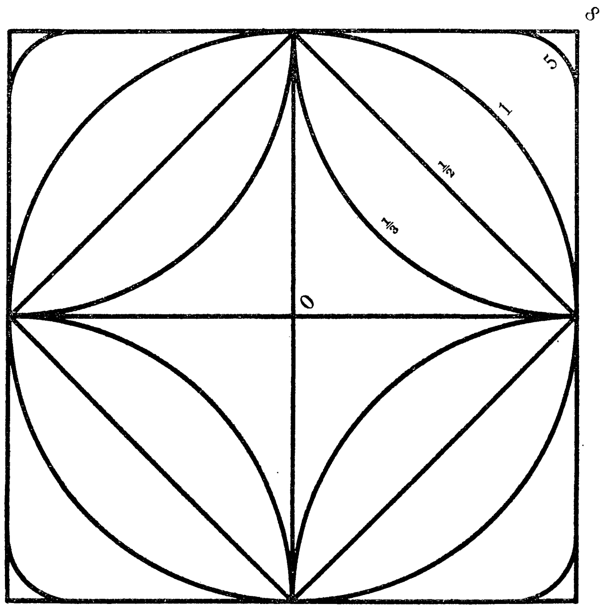

FIG. Ib. Locus of $x^{2n}+y^{2n}=r^{2n}$ for $n=0, \frac{1}{3}, \frac{1}{2}, 1, 5$ and ∞.

Using similar methods, we can show that the volume of a pseudo-cube is

$$V = \frac{8r^3\{\Gamma(1/2n + 1)\}^3}{\Gamma(3/2n + 1)} . \tag{5}$$

When $n \geqq 5$ the perimeter and lateral area of a pseudo-cube can be expressed respectively by the following approximations:

$$P \doteq \frac{8r\{\Gamma(1/2n + 1)\}^2}{\Gamma(1/n + 1)}, \qquad n \geqq 5 \tag{6}$$

$$S \doteq \frac{24r^2\{\Gamma(1/2n + 1)\}^3}{\Gamma(3/2n + 1)}, \qquad n \geqq 5. \tag{7}$$

Formula (6) is obtained by evaluating dA/dr from (4). Thus

$$\frac{dA}{dr} = \frac{\partial A}{\partial r} + \frac{\partial A}{\partial n}\frac{dn}{dr}, \qquad \frac{\partial A}{\partial r} = \frac{8r\{\Gamma(1/2n + 1)\}^2}{\Gamma(1/n + 1)},$$

and $dn/dr = 0$ when $n = 1$ or $n \to \infty$; that is, $n \neq f(r)$ for concentric circles and geometrical squares. This explains why formulas (6) and (7) become exact for the circle and the geometrical square.

EXAMPLE 1. The table below lists a few well-known curves which are particular and limiting cases of the general expression.

a	n	i	Curve	Example
$a_1 = a_2 = a_3$	∞	1, 2, 3	Cube	$\lim_{n\to\infty}\left[\left(\frac{x}{2}\right)^{2n}+\left(\frac{y}{2}\right)^{2n}+\left(\frac{z}{2}\right)^{2n}\right]=1$
$a_1 = a_2$	∞	1, 2	Square	$\lim_{n\to\infty}\left[\left(\frac{x}{2}\right)^{2n}+\left(\frac{y}{2}\right)^{2n}\right]=1$
$a_1 = a_2 = a_3$	1	1, 2, 3	Sphere	$\left[\left(\frac{x}{2}\right)^{2}+\left(\frac{y}{2}\right)^{2}+\left(\frac{z}{2}\right)^{2}\right]=1$
$a_1 = a_2$	1	1, 2	Circle	$\left[\left(\frac{x}{2}\right)^{2}+\left(\frac{y}{2}\right)^{2}\right]=1$
a_1, a_2, real	1/2	1, 2	Rectangular rhombus	$\frac{\lvert x\rvert}{2}+\frac{\lvert y\rvert}{3}=1$
$a_1 \neq a_2 \neq a_3$	∞	1, 2, 3	Parallelepiped	$\lim_{n\to\infty}\left[\left(\frac{x}{2}\right)^{2n}+\left(\frac{y}{3}\right)^{2n}+\left(\frac{z}{4}\right)^{2n}\right]=1$
$a_1 \neq a_2$	∞	1, 2	Rectangle	$\lim_{n\to\infty}\left[\left(\frac{x}{2}\right)^{2n}+\left(\frac{y}{3}\right)^{2n}\right]=1$
$a_1 \neq a_2$	1	1, 2	Ellipse	$\left[\left(\frac{x}{2}\right)^{2}+\left(\frac{y}{3}\right)^{2}\right]=1$
$a_1 \neq a_2 \neq a_3$	1	1, 2, 3	Ellipsoid	$\left[\left(\frac{x}{2}\right)^{2}+\left(\frac{y}{3}\right)^{2}+\left(\frac{z}{4}\right)^{2}\right]=1$
$a_1 = a_2$	1/2	1, 2	Square rhombus	$(x^2)^{1/2}+(y^2)^{1/2}=1$

EXAMPLE 2. A pyramid can be represented in cartesian coordinates by the equation $x^{2n}+y^{2n}=cz$, c being a constant. The left hand side of this equation contains a parameter n. When $n=1$, we get the equation of a paraboloid of revolution; when $n\to\infty$, we get the equation of the geometrical pyramid. When $z=z_1$, the area of the base is

$$\frac{4r^2\{\Gamma(1/2n+1)\}^2}{\Gamma(1/n+1)}$$

and $cz_1=r^{2n}=$constant. A satisfactory approach to the geometrical pyramid is obtained with $n \geqq 5$. The volume of this pyramid is $(Bz_1)/3$ where B is the area of the base and z_1 is the height; hence

$$V = \frac{4r^2 z_1 \{\Gamma(1/2n + 1)\}^2}{3\Gamma(1/n + 1)}.$$

When $n \to \infty$, $V = 4r^2 z_1/3$ which is the volume of the geometrical pyramid. When $n = 1$, $V = \pi r^2 z_1/3$ which is the volume of a cone.

EXAMPLE 3. Show that the expression $x^N + y^N = z^N$ can have solutions as close as one wishes to positive integer values when $N \to \infty$.

Consider the cartesian locus of the expression $\lim_{n\to\infty} [x^{2n} + y^{2n} = z^{2n}]$. Let us put z = constant and n a positive integer parameter. If we divide both sides by z^{2n} we get

$$\lim_{n \to \infty} \left[\left(\frac{x}{z} \right)^{2n} + \left(\frac{y}{z} \right)^{2n} \right] = 1.$$

This locus contains many positive integer values, in particular the point (2, 1, 2), which belongs to the locus because

$$\lim_{n \to \infty} \left[\left(\frac{2}{2} \right)^{2n} + \left(\frac{1}{2} \right)^{2n} \right] = 1.$$

Hence, when $2n = N$, n a positive integer, the expression $x^N + y^N = z^N$ can have solutions that are as close as one wishes to positive integer values.

References

1. A. E. Taylor, Introduction to functional analysis, Wiley, New York, 1958, pp. 87–94.
2. G. H. Hardy, A course of pure mathematics, Cambridge, New York, 1958, p. 47.
3. J. Allard and J. Bazinet, Remarques Sur Quelques Figures Géométriques, Revue de l'Université de Sherbrooke, 2, No. 2. 1962.

A SIMPLE MATRIX INVERSION PROCEDURE?

WILLIAM SQUIRE, West Virginia University

It is well known that solving a particular set of linear equations, $AX = C$, where A is a square matrix and X and C column matrices, is much easier than inverting A by conventional methods. Therefore, I had dreams of glory when I "discovered" the following scheme.

Since $A^{-1}C = X$, it is easily shown that $A^{-1} = XR$ where $R = C^T(CC^T)^{-1}$, C^T being the transpose of C. It appeared that once a standard set of column matrices C and the corresponding row matrices R were tabulated, inverting a matrix would be reduced to the simpler problem of solving a single system of linear equations and a matrix multiplication. Visions of tables of Squire Matrices flashed before me, until I tried to compute them. Then the bubble burst! Why?

Unfortunately, the determinant of the matrix CC^T must vanish. Therefore, $(CC^T)^{-1}$ and R do not exist.

obtained with $n \geqq 5$. The volume of this pyramid is $(Bz_1)/3$ where B is the area of the base and z_1 is the height; hence

$$V = \frac{4r^2 z_1 \{\Gamma(1/2n + 1)\}^2}{3\Gamma(1/n + 1)}.$$

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Unfortunately, the determinant of the matrix CC^T must vanish. Therefore, $(CC^T)^{-1}$ and R do not exist.

ON THE n-th DERIVATIVE OF A DETERMINANT OF THE j-th ORDER

JOHN G. CHRISTIANO, Northern Illinois University and
JAMES E. HALL, University of Wisconsin

An expression due to Leibniz not commonly seen in today's beginning calculus is the formula for the n-th derivative of a product of two differentiable functions,

$$\frac{d^n}{dx^n}(uv) = \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} \frac{d^j}{dx^j} u \frac{d^{n-j}}{dx^{n-j}} v. \tag{1}$$

The purpose of this note is to extend this expression to products of any number of functions by symbolic methods and to use the result in finding the n-th derivative of a determinant of the j-th order whose elements are differentiable functions.

Set

$$\frac{d^n}{dx^n} f = f\overset{n}{\Delta}_f, \quad \Delta_f = \overset{1}{\Delta}_f, \quad \Delta_{u_1,\cdots,u_n} = \Delta_{u_1} + \cdots + \Delta_{u_n}. \tag{2}$$

We interpret Δ_f^0 as the identity operator on f. The operator Δ_f^n acts only on the function f, so that $(uv)\Delta_u^n = (u\Delta_u^n)v$ and $(uv)\Delta_v^n = u(v\Delta_v^n)$. No meaning need be attached to $v\Delta_u^n$ since we shall never apply Δ_u^n to a function not containing u as a factor. Note that $(uv)(\Delta_u + \Delta_v) = (uv)\Delta_u + (uv)\Delta_v$ and that $\Delta_u\Delta_v = \Delta_v\Delta_u$ trivially, because Δ_f operates specifically on the function f.

Using equation (2), we can write equation (1) as

$$\frac{d^n}{dx^n}(uv) = uv\overset{n}{\Delta}_{u,v} = uv(\Delta_u + \Delta_v)^n, \qquad n = 1, 2, 3, \cdots. \tag{3}$$

Similarly,

$$\frac{d^n}{dx^n}(uvw) = uvw\overset{n}{\Delta}_{u,v,w} = uvw(\Delta_u + \Delta_v + \Delta_w)^n \tag{4}$$

and in general

$$\frac{d^n}{dx^n}(u_1u_2u_3 \cdots u_j) = \left(\prod_{\alpha=1}^{j} u_\alpha\right)\left(\sum_{\alpha=1}^{j} \Delta_{u_\alpha}\right)^n. \tag{5}$$

Example 1. Find the second derivative of a product of three functions.

$$\begin{aligned}\frac{d^2}{dx^2}(u_1u_2u_3) &= u_1u_2u_3(\Delta_{u_1} + \Delta_{u_2} + \Delta_{u_3})^2 \\ &= u_1u_2u_3[\overset{2}{\Delta}_{u_1} + \overset{2}{\Delta}_{u_2} + \overset{2}{\Delta}_{u_3} + 2(\Delta_{u_1}\Delta_{u_2} + \Delta_{u_1}\Delta_{u_3} + \Delta_{u_2}\Delta_{u_3})].\end{aligned}$$

Now apply equation (2) and get

$$\frac{d^2}{dx^2}(u_1u_2u_3) = u_2u_3\frac{d^2}{dx^2}u_1 + u_1u_3\frac{d^2}{dx^2}u_2 + u_1u_2\frac{d^2}{dx^2}u_3 + 2u_3\frac{d}{dx}u_1\frac{d}{dx}u_2 + 2u_2\frac{d}{dx}u_1\frac{d}{dx}u_3 + 2u_1\frac{d}{dx}u_2\frac{d}{dx}u_3. \tag{6}$$

If, now, $\prod_{\alpha=1}^{j} u_\alpha$ represents one of the products in the expansion of a determinant of the jth order, equation (5) can be used to write the nth derivative of such a determinant. Let

$$D = \begin{vmatrix} f_{11} & f_{12} \cdots f_{1j} \\ f_{21} & f_{22} \cdots f_{2j} \\ \cdot & \cdot \ \cdot \cdot \cdot \cdot \\ f_{j1} & f_{j2} \cdots f_{jj} \end{vmatrix}, \tag{7}$$

where the f_{ik} are differentiable functions of the independent variable x. Then

$$D = \sum_{\phi\in S} \operatorname{sgn}\phi\, f_{1\phi(1)}f_{2\phi(2)}\cdots f_{j\phi(j)}, \tag{8}$$

where S is the set of permutations ϕ of the j subscripts and sgn ϕ is $+1$ or -1 according as ϕ is an even or an odd permutation. Then

$$\frac{d^n}{dx^n}D = \sum_{\phi\in S} \operatorname{sgn}\phi\,\frac{d^n}{dx^n}(f_{1\phi(1)}f_{2\phi(2)}\cdots f_{j\phi(j)}),$$

or, using (5),

$$\begin{aligned} \frac{d^n}{dx^n}D &= \sum_{\phi\in S}\operatorname{sgn}\phi\left\{\left[\prod_{\alpha=1}^{j} f_{\alpha\phi(\alpha)}\right]\left[\sum_{\alpha=1}^{j}\Delta_{f_{\alpha\phi(\alpha)}}\right]^n\right\} \\ &= \sum_{\phi\in S}\operatorname{sgn}\phi\left\{\left[\prod_{\alpha=1}^{j} f_{\alpha\phi(\alpha)}\right]\left[\Delta_{f_{1\phi(1)}}+\Delta_{f_{2\phi(2)}}+\cdots+\Delta_{f_{j\phi(j)}}\right]^n\right\} \\ &= \sum_{\phi\in S}\operatorname{sgn}\phi\left\{\left[\prod_{\alpha=1}^{j} f_{\alpha\phi(\alpha)}\right]\left[\sum\frac{n!}{\eta_1!\eta_2!\cdots\eta_j!}\Delta_{f_{1\phi(1)}}^{\eta_1}\cdots\Delta_{f_{j\phi(j)}}^{\eta_j}\right]\right\}, \end{aligned} \tag{9}$$

the inner sum being over all positive integral values of the η's subject to the restriction $\eta_1+\eta_2+\cdots+\eta_j=n$. Since the sums and products are finite, we have

$$\begin{aligned} \frac{d^n}{dx^n}D &= \sum\frac{n!}{\eta_1!\eta_2!\cdots\eta_j!}\left\{\sum_{\phi\in S}\operatorname{sgn}\phi\left[\prod_{\alpha=1}^{j} f_{\alpha\phi(\alpha)}\Delta_{f_{\alpha\phi(\alpha)}}^{\eta_\alpha}\right]\right\} \\ &= \sum\frac{n!}{\eta_1!\eta_2!\cdots\eta_j!}\begin{vmatrix} f_{11}^{(\eta_1)} & f_{12}^{(\eta_1)} & \cdots & f_{1j}^{(\eta_1)} \\ f_{21}^{(\eta_2)} & f_{22}^{(\eta_2)} & \cdots & f_{2j}^{(\eta_2)} \\ \cdot & \cdot & \cdots & \cdot \\ f_{j1}^{(\eta_j)} & f_{j2}^{(\eta_j)} & \cdots & f_{jj}^{(\eta_j)} \end{vmatrix}, \end{aligned} \tag{10}$$

where $f_{mk}^{(\eta m)}=f\Delta_{fmk}^{\eta_m}$ and $\sum\eta_k=n$. Hence the expansion is a sum of determinants which correspond to the terms of the multinomial expansion, i.e., the expansion of $(x_1+x_2+\cdots+x_j)^n$. In each of these determinants, the sum of the orders of the derivatives in any column equals n and the orders in any row are the same since each value of α gives the order η_α for the derivative for each $\phi(\alpha)$, i.e., for each column. The terms of the expansion correspond to the possible ways of choosing the η's consistent with $\sum \eta_k = n$.

Example 2. Write the third derivative of the third order determinant

$$D = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix},$$

i.e., find $(d^3/dx^3)D$. This is a sum of determinants determined by η_1, η_2, η_3 such that $\eta_1+\eta_2+\eta_3=3$. There are ten such combinations which are listed below with the corresponding coefficient for each, $3!/(\eta_1!\eta_2!\eta_3!)$.

η_1	η_2	η_3	$\dfrac{3!}{\eta_1!\eta_2!\eta_3!}$
3	0	0	1
0	3	0	1
0	0	3	1
2	1	0	3
2	0	1	3
1	2	0	3
0	2	1	3
1	0	2	3
0	1	2	3
1	1	1	6

Hence we write, where a prime indicates derivative with respect to x,

$$\begin{aligned}
\frac{d^3}{dx^3} D = &\begin{vmatrix} f_{11}''' & f_{12}''' & f_{13}''' \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} + \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21}''' & f_{22}''' & f_{23}''' \\ f_{31} & f_{32} & f_{33} \end{vmatrix} + \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31}''' & f_{32}''' & f_{33}''' \end{vmatrix} \\
&+ 3\begin{vmatrix} f_{11}'' & f_{12}'' & f_{13}'' \\ f_{21}' & f_{22}' & f_{23}' \\ f_{31} & f_{32} & f_{33} \end{vmatrix} + 3\begin{vmatrix} f_{11}'' & f_{12}'' & f_{13}'' \\ f_{21} & f_{22} & f_{23} \\ f_{31}' & f_{32}' & f_{33}' \end{vmatrix} + 3\begin{vmatrix} f_{11}' & f_{12}' & f_{13}' \\ f_{21}'' & f_{22}'' & f_{23}'' \\ f_{31} & f_{32} & f_{33} \end{vmatrix} \\
&+ 3\begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21}'' & f_{22}'' & f_{23}'' \\ f_{31}' & f_{32}' & f_{33}' \end{vmatrix} + 3\begin{vmatrix} f_{11}' & f_{12}' & f_{13}' \\ f_{21} & f_{22} & f_{23} \\ f_{31}'' & f_{32}'' & f_{33}'' \end{vmatrix} + 3\begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21}' & f_{22}' & f_{23}' \\ f_{31}'' & f_{32}'' & f_{33}'' \end{vmatrix} \\
&+ 6\begin{vmatrix} f_{11}' & f_{12}' & f_{13}' \\ f_{21}' & f_{22}' & f_{23}' \\ f_{31}' & f_{32}' & f_{33}' \end{vmatrix}.
\end{aligned} \tag{11}$$

NOTE ON CONSECUTIVE INTEGERS WHOSE SUM OF SQUARES IS A PERFECT SQUARE

STANTON PHILIPP, Seal Beach, California

In [1], Brother U. Alfred considered the problem of determining for which positive integers n is the equation

$$nx^2 + n(n-1)x + \frac{(n-1)n(2n-1)}{6} = z^2 \tag{1}$$

solvable in positive integers x and z. The following problems were left for solution in [1]:

(a) To determine whether or not (1) has positive integer solutions for the following values of n: 107, 193, 227, 275, 457.

(b) Given that there is at least one positive integer solution of (1), to determine whether the total number of such solutions is finite or infinite.

(c) To determine a recursion formula for the infinite set of positive integer solutions of (1), in case infinitely many such solutions exist.

This note is concerned with the solution of the above problems. Problem (a) is solved in the third section of this note; the result is that there exist positive integer solutions of (1) for $n=107$, $n=193$, $n=457$, while there are no integer solutions of (1) for $n=227$, $n=275$. In the first and second sections, we answer (b) as follows: given a positive integer solution of (1), there are finitely or infinitely many such solutions according as n is or is not a perfect square. In the course of proving Theorem 1, we obtain a partial solution of (c); that is, when n is not a perfect square, we obtain an infinite set of solutions corresponding to a particular one, although we do not necessarily obtain all solutions.

1. *Remarks on equation* (1). Under the substitutions $u=2z$, $y=2x+n-1$, equation (1) becomes

$$u^2 - ny^2 = \frac{n(n^2-1)}{3}\,. \tag{2}$$

If n is a perfect square, the number of solutions of (2), and hence of (1), is at most finite, and the solutions of (1) (if any) are easily obtained. If n is given and n is not a perfect square, there is a procedure, due to Lagrange, for determining systematically all solutions of (2) in integers. A simple account of this procedure and a reference to Lagrange's work are given in [2]. Suffice it to say here that the calculations involved in Lagrange's method are often formidable.

2. THEOREM 1. *If n is not a perfect square and if* (1) *has a positive integer solution* (x, z), *then* (1) *has infinitely many positive integer solutions.*

Proof. Let (a, b) be a positive integer solution of (1). Then (2) has the solution

$$u_0 = 2b, \qquad y_0 = 2a + n - 1. \tag{3}$$

The Pell equation $v^2-nt^2=1$ has infinitely many positive integer solutions (v, t). For each such (v, t), we have $(v^2-nt^2)(u_0^2-ny_0^2)=\{n(n^2-1)/3\}$, or $(vu_0+nty_0)^2-n(tu_0+vy_0)^2=\{n(n^2-1)/3\}$, so that $u=vu_0+nty_0$, $y=tu_0+vy_0$ is a solution of (2). Then

$$z=\frac{vu_0+nty_0}{2},\qquad x=\frac{tu_0+vy_0+1-n}{2} \tag{4}$$

is a solution of (1). It remains to show that x and z, as given by (4), are positive integers. Clearly, x and z will be positive for sufficiently large v and t. If n is odd, then x and z are integers by (3) and (4). If n is even, then x and z are integers by (3), (4), and the assumption $v^2-nt^2=1$.

3. *The cases* $n=107$, $n=193$, $n=227$, $n=275$ *and* $n=457$. In this section, all letters introduced will denote integers.

LEMMA 1. *If $p=4k+3$ is prime and if $p\mid(x^2+y^2)$, then $p\mid x$ and $p\mid y$.*

As Lemma 1 is equivalent to a well-known theorem on the number of representations of an integer as a sum of two squares (see [**3**]), we omit proof.

We proceed to apply Lemma 1 to dispose of the cases $n=107$ and $n=227$.

If $n=107$, (1) becomes

$$107x^2+107\cdot 106x+107\cdot 53\cdot 71=z^2. \tag{5}$$

Since the prime 107 must divide z^2, we may write $z=107u$. Then $x^2+106x+53\cdot 71=107u^2$. Completing the square, we have $(x+53)^2+9\cdot 106=107u^2$. Letting $y=x+53$, we have $y^2+9\cdot 106=107u^2$. Then $y^2+u^2+9\cdot 106=108u^2$. Hence $3\mid(y^2+u^2)$, so by Lemma 1 we may write $y=3s$, $u=3r$. Then

$$s^2+106=107r^2. \tag{6}$$

It follows that $106\mid(s^2-r^2)$. Noting that $106=2\cdot 53$, 53 is prime, and that $s^2-r^2=(s-r)(s+r)$, we see that either $106\mid(s-r)$ or $106\mid(s+r)$. In either event we may write $r^2=(s-106t)^2$. Substituting this last expression in (6) we obtain $s^2+106=107\ (s-106t)^2$, or $s^2-2\cdot 107st+107\cdot 106t^2=1$. Completing the square, we have $(s-107t)^2-107t^2=1$. Let $v=s-107t$. Now we have reduced (5) to the Pell equation

$$v^2-107t^2=1. \tag{7}$$

We can easily retrace our steps and find that

$$x=3(v+107t)-53,\qquad z=\pm\ 3\cdot 107(v+t). \tag{8}$$

Therefore (5) has infinitely many positive integer solutions (x, z). The minimal positive solution of (7) is easily found by expanding $\sqrt{107}$ as a simple continued fraction; it is $v=962$, $t=93$. Thus, by (8), one solution of (5) is $x=32686$, $z=338655$.

In case $n=227$, we must deal with

$$227x^2+227\cdot 226x+227\cdot 113\cdot 151=z^2. \tag{9}$$

As in the above, we may write $z=227u$, complete the square, and let $y=x+113$;

we obtain $y^2+19\cdot 226=227u^2$. Then $19|(y^2+u^2)$, so by Lemma 1 we may write $y=19s$, $u=19r$. It follows that

(10) $$19s^2+226=227\cdot 19r^2.$$

But (10) is absurd, since 19 does not divide 226. Hence, (9) has no solution in integers x and z.

For the case $n=193$, notice that $(193\cdot 12)^2-193\cdot 124^2=\{193(193^2-1)/3\}$. Hence, from the proof of Theorem 1, we infer that (1) has infinitely many positive integer solutions for $n=193$.

The case $n=275$ can be handled by methods similar to those in [1]. In this case, (1) becomes

(11) $$275x^2+275\cdot 274x+275\cdot 137\cdot 183=z^2.$$

Since 5 and 11 are factors of 275 and since $275|z^2$, we may write $z=55u$. Then $x^2+274x+137\cdot 183=11u^2$. Completing the square and letting $y=x+137$, we obtain

(12) $$y^2+137\cdot 46=11u^2.$$

It follows from (12) that y and u must be of the same parity; moreover, they cannot both be even because 4 does not divide $137\cdot 46$. Thus we may write $y=2a+1$, $u=2b+1$ in (12). We obtain

(13) $$(a^2+a)+1573=11(b^2+b).$$

Since a^2+a and b^2+b are both even, (13) is absurd. It follows that (11) has no solution in integers x and z.

If $n=457$, (1) becomes

(14) $$457x^2+457\cdot 456x+457\cdot 76\cdot 913=z^2.$$

We may write $z=457u$, and let $y=x+228$. Then

(15) $$y^2-457u^2=(-19)(916).$$

From the simple continued fraction for $\sqrt{457}$, we can calculate that $(583531623)^2-457(27296458)^2=-19$; by direct trial, we find that $(5644)^2-457(264)^2=916$. It follows that (15) has the solution

$$y=5644\cdot 583531623+457\cdot 264\cdot 27296458, \qquad u=264\cdot 583531623+5644\cdot 27296458.$$

It is now immediate that (14) has positive integer solutions.

It is my pleasure to thank the referee for his valuable suggestions for improving this paper.

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A HYPERBOLIC PROPOSAL

William R. Ransom, Tufts University

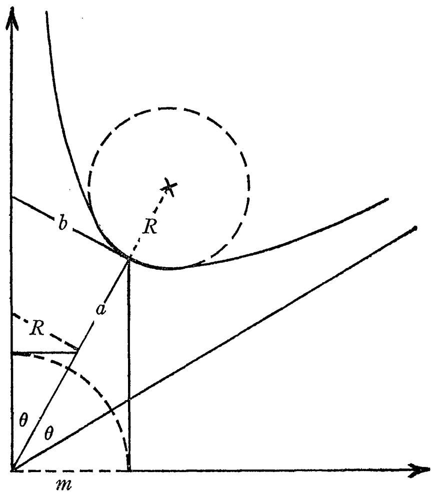
The simplicity of the equation $y=m^2/x$ for equilateral hyperbolas, giving as it does a single-valued function for the ordinate, suggests that for an oblique hyperbola the Y-scale be retained as an asymptote.

If the asymptotes form an angle of 2θ in the first quadrant, the equation

$$y = (m^2 + x^2 \cos 2\theta) \div x \sin 2\theta$$

describes the hyperbola. Its vertex is at $(m, m \cot \theta)$. Its semiaxes are $a=m/\sin\theta$ and $b=m/\cos\theta$, and the radius of curvature at the vertex is $R=m\tan\theta \div \cos\theta$. All of these are readily constructed.

Some of the trigonometric transformations involved in this consideration are unusual and interesting.



ON FIBONACCI SEQUENCES AND A GEOMETRICAL PARADOX

Santosh Kumar, Armament Research and Development Establishment, Kirkee, Poona-3, India

1. In dealing with the geometrical paradox connected with Fibonacci sequences, Horadam [1], [2] has

a. extended the paradox to all sets of three consecutive Fibonacci numbers;

b. established a formula for θ_n, the acute angle between the sides of the small parallelogram which gives rise to the paradox; and

c. generalized the paradox in terms of generalized Fibonacci sequences.

As is well known, a Fibonacci sequence (fn) is defined by the relations:

$$F_1 = 1 = F_2, \qquad F_n = F_{n-1} + F_{n-2}, \qquad n \geqq 3.$$

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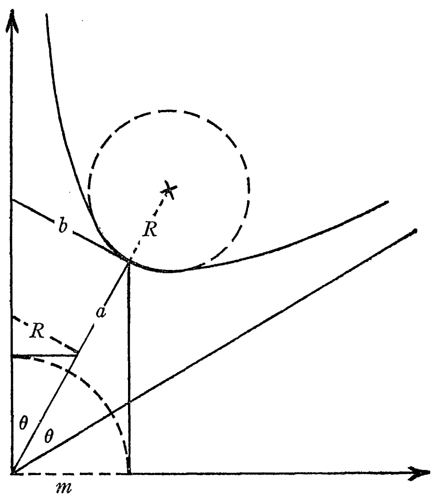

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As is well known, a Fibonacci sequence (fn) is defined by the relations:

$$F_1 = 1 = F_2, \qquad F_n = F_{n-1} + F_{n-2}, \qquad n \geqq 3.$$

The relation which gives rise to the paradox is $|F_n^2 - F_{n-1}F_{n+1}| = 1$.

We take a square of side F_n, divide it into four parts which can be rearranged so as to look as if they form a rectangle with sides F_{n-1} and F_{n+1}. A unit area thus appears to have been lost or gained. This is the paradox. What happens actually is that the parts either overlap or leave a small parallelogram in between. Here, we give an alternative method of arranging the parts which maintains the paradox. Our idea easily admits of extension to generalized Fibonacci sequences and θ_n can be found as readily as in [1] and [2]. The paradox does not exist for $n<4$. An interesting phenomenon is observed when the division of the square is repeated and rearrangements of the parts are done alternately according to Horadam and the method given here.

The method of division of the square on F_n, $n>4$, and the two ways of rearranging the parts are exhibited in the following simple diagrams:

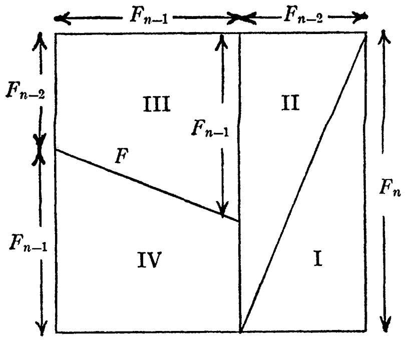

FIG. 1.

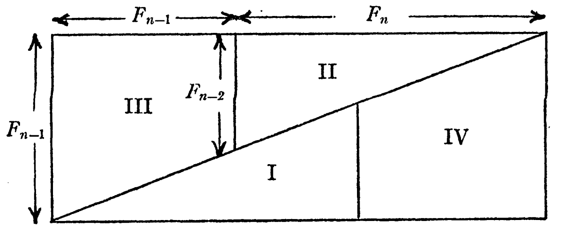

FIG. 2.

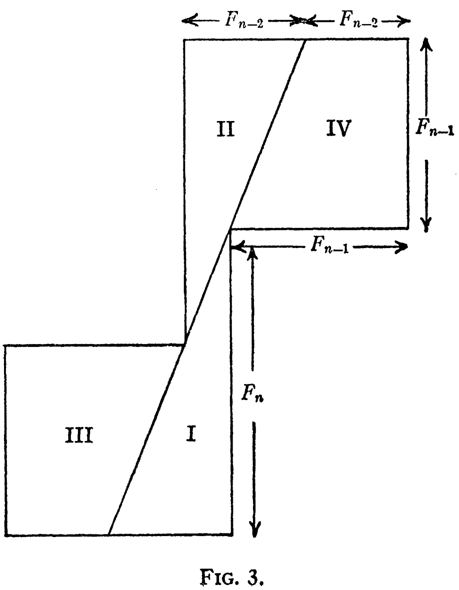

FIG. 3.

Apparently

$$\text{Area in Fig. 2} = F_{n-1}(F_n + F_{n-1}) = F_{n-1}F_{n+1}.$$

$$\begin{aligned}\text{Area in Fig. 3} &= 2F_{n-1}^2 + (F_n + F_{n-1})(2F_{n-2} - F_{n-1}) \\ &= 2F_{n-1}^2 + F_{n+1}F_{n-4} \\ &= F_{n-2}\cdot F_{n+2}.\end{aligned}$$

We notice that $|F_n^2 - F_{n-2}F_{n+2}| = 1$.

The arrangements of Fig. 2 and Fig. 3 will be called arrangement A and arrangement B respectively.

We can subdivide the square on the side of length F_n and rearrange it either according to A or according to B. By arranging in this way we get two squares on the side of length F_{n-1}. These squares can again be subdivided and rearranged according to either B or A respectively. Similar successive rearrangements, e.g. "A" followed by "A" are out of the question because of overlapping. In the second subdivision we have four squares on the side of length F_{n-2}. However, only the squares at the end can be subdivided further. The two intermediate squares cannot be subdivided due to overlapping. The successive subdivision can be repeated until we get the square on the side of length F_4. This cannot be carried on further as the paradox does not exist for $n<3$. The results may be summarized in the following form:

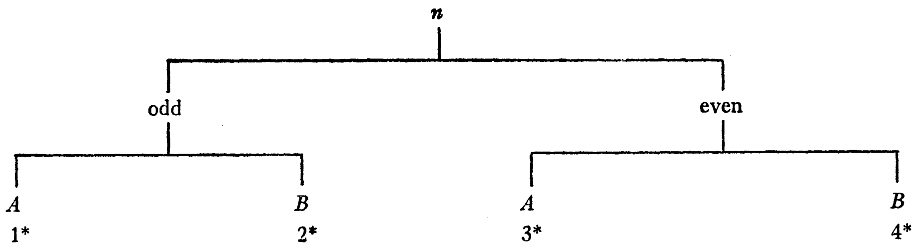

FIG. 4.

1* and 4* give decrease in area; 2* and 3* give increase in area.

From the nature of the paradox, we notice that if the first selection is made so as to obtain an increased area, the area will go on increasing as a result of successive subdivision. We cannot decrease the area at any intermediate stage. A similar result holds for the case when we have the area originally decreased. Obtaining a formula for the total increase or decrease is an interesting problem, which deserves attention.

The author wishes to express his gratitude to Dr. J. N. Sekhri for his valuable suggestions.

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CIRCLE ASSOCIATE OF A GIVEN POINT

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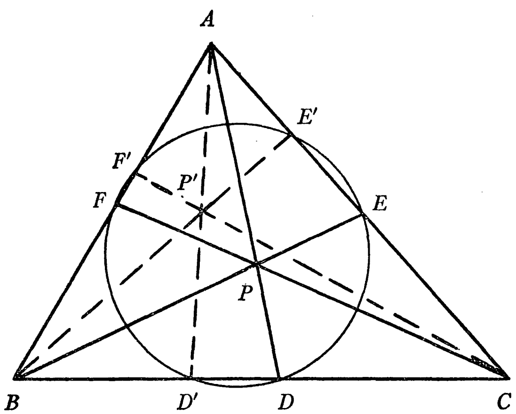
Let P be any point in the plane of triangle ABC and let DEF be its cevian triangle. Circle DEF is constructed to meet sides BC, CA, AB again at points D', E', F' respectively. Points E, E', F', F are cyclic whence $AF\cdot F'A = EA\cdot AE'$ or $AF/EA = AE'/F'A$. In similar fashion $BD/FB = BF'/D'B$ and $CE/DC = CD'/E'C$. So $(AF/EA)\cdot(BD/FB)\cdot(CE/DC) = (AE'/F'A)\cdot(BF'/D'B)\cdot(CD'/E'C)$ or $(BD/DC)\cdot(CE/EA)\cdot(AF/FB) = (CD'/D'B)\cdot(BF'/F'A)\cdot(AE'/E'C)$. From Ceva's theorem it is known that $(BD/DC)\cdot(CE/EA)\cdot(AF/FB) = 1$ and consequently $(CD'/D'B)\cdot(BF'/F'A)\cdot(AE'/E'C) = 1$ which means that rays AD', BE', CF' are concurrent at a point P'. This well-known result then follows:

THEOREM 1. *Let P be any point in the plane of triangle ABC, with DEF its cevian triangle. Construct circle DEF to meet sides BC, CA, AB again at points D', E', F' respectively. Then the rays AD', BE', CF' are concurrent.*



The point P' thus determined will be called the *circle associate* of the point P with respect to triangle ABC. We attempt to find ratio values for $BD'/D'C$, $CE'/E'A$, $AF'/F'B$ in terms of ratios BD/DC, CE/EA, AF/FB and sides a, b, c of triangle ABC. (It is understood that a, b, c represent the sides BC, CA, AB respectively of the given triangle.)

Since points E, E', F', F are cyclic, we may write $AF\cdot AF' = AE\cdot AE'$, or $(AF/AE)\cdot(AF'/AE') = 1$. Now

$$\frac{AF}{AE} = \frac{FA}{EA} = \frac{c}{b}\cdot\frac{b}{EA}\cdot\frac{FA}{c} = \frac{c}{b}\cdot\frac{\dfrac{b}{EA}}{\dfrac{c}{FA}} = \frac{c}{b}\left[\frac{\dfrac{CE+EA}{EA}}{\dfrac{BF+FA}{FA}}\right] = \frac{c}{b}\left[\frac{\dfrac{CE}{EA}+1}{\dfrac{BF}{FA}+1}\right].$$

In similar fashion

$$\frac{AF'}{AE'}=\frac{c}{b}\left\{\frac{\dfrac{CE'}{E'A}+1}{\dfrac{BF'}{F'A}+1}\right\}.$$

The equation $(AF/AE)\cdot(AF'/AE')=1$ then becomes

$$\frac{c^2}{b^2}\left\{\frac{\dfrac{CE}{EA}+1}{\dfrac{BF}{FA}+1}\right\}\left\{\frac{\dfrac{CE'}{E'A}+1}{\dfrac{BF'}{F'A}+1}\right\}=1$$

from which it follows that

$$\frac{BF'}{F'A}=\frac{c^2}{b^2}\left\{\frac{\dfrac{CE}{EA}+1}{\dfrac{BF}{FA}+1}\right\}\left(\frac{CE'}{E'A}+1\right)-1.$$

Applying the same procedure with respect to the vertex B and the cyclic points F', F, D', D, we obtain

$$\frac{a^2}{c^2}\left\{\frac{\dfrac{AF}{FB}+1}{\dfrac{CD}{DB}+1}\right\}\left\{\frac{\dfrac{AF'}{F'B}+1}{\dfrac{CD'}{D'B}+1}\right\}=1.$$

Since $(BD'/D'C)\cdot(CE'/E'A)\cdot(AF'/F'B)=1$, it is possible to replace $CD'/D'B$ by $(CE'/E'A)\cdot(AF'/F'B)$ in the preceding equation. When this is done, the resulting equation may be solved for $BF'/F'A$. A computation of some length yields

$$\frac{BF'}{F'A}=\frac{\dfrac{a^2}{c^2}\left(\dfrac{AF}{FB}+1\right)-\dfrac{CE'}{E'A}\left(\dfrac{CD}{DB}+1\right)}{\left(\dfrac{CD}{DB}+1\right)-\dfrac{a^2}{c^2}\left(\dfrac{AF}{FB}+1\right)}.$$

This value of $BF'/F'A$ may be placed equal to that obtained at the end of the preceding paragraph. The equation thus secured may be solved for $CE'/E'A$. After another lengthy computation and simplification it is found that

$$\frac{CE'}{E'A}=\frac{b^2\left(\dfrac{CD}{DB}+1\right)\left(\dfrac{BF}{FA}+1\right)-c^2\left(\dfrac{CE}{EA}+1\right)\left(\dfrac{CD}{DB}+1\right)+a^2\left(\dfrac{CE}{EA}+1\right)\left(\dfrac{AF}{FB}+1\right)}{b^2\left(\dfrac{CD}{DB}+1\right)\left(\dfrac{BF}{FA}+1\right)+c^2\left(\dfrac{CE}{EA}+1\right)\left(\dfrac{CD}{DB}+1\right)-a^2\left(\dfrac{CE}{EA}+1\right)\left(\dfrac{AF}{FB}+1\right)}.$$

In similar fashion the values of $AF'/F'B$ and $BD'/D'C$ exhibited below are determined.

THEOREM 2. *Let P be any point in the plane of triangle ABC, with DEF its cevian triangle. Construct circle DEF to meet sides BC, CA, AB again at points D', E', F' respectively. Then the rays AD', BE', CF' are concurrent and*

$$\frac{BD'}{D'C}=\frac{a^2\left(\frac{BF}{FA}+1\right)\left(\frac{AE}{EC}+1\right)-b^2\left(\frac{BD}{DC}+1\right)\left(\frac{BF}{FA}+1\right)+c^2\left(\frac{BD}{DC}+1\right)\left(\frac{CE}{EA}+1\right)}{a^2\left(\frac{BF}{FA}+1\right)\left(\frac{AE}{EC}+1\right)+b^2\left(\frac{BD}{DC}+1\right)\left(\frac{BF}{FA}+1\right)-c^2\left(\frac{BD}{DC}+1\right)\left(\frac{CE}{EA}+1\right)},$$

$$\frac{CE'}{E'A}=\frac{b^2\left(\frac{CD}{DB}+1\right)\left(\frac{BF}{FA}+1\right)-c^2\left(\frac{CE}{EA}+1\right)\left(\frac{CD}{DB}+1\right)+a^2\left(\frac{CE}{EA}+1\right)\left(\frac{AF}{FB}+1\right)}{b^2\left(\frac{CD}{DB}+1\right)\left(\frac{BF}{FA}+1\right)+c^2\left(\frac{CE}{EA}+1\right)\left(\frac{CD}{DB}+1\right)-a^2\left(\frac{CE}{EA}+1\right)\left(\frac{AF}{FB}+1\right)},$$

$$\frac{AF'}{F'B}=\frac{c^2\left(\frac{AE}{EC}+1\right)\left(\frac{CD}{DB}+1\right)-a^2\left(\frac{AF}{FB}+1\right)\left(\frac{AE}{EC}+1\right)+b^2\left(\frac{AF}{FB}+1\right)\left(\frac{BD}{DC}+1\right)}{c^2\left(\frac{AE}{EC}+1\right)\left(\frac{CD}{DB}+1\right)+a^2\left(\frac{AF}{FB}+1\right)\left(\frac{AE}{EC}+1\right)-b^2\left(\frac{AF}{FB}+1\right)\left(\frac{BD}{DC}+1\right)}.$$

In using the theorem it must be remembered that the segments involved are to be considered as directed quantities. That is, if D lies between B and C, then BD/DC is considered positive. If D lies on BC extended, then BD/DC must be considered negative. Similarly the other ratios involved in the results given are signed ratios.

If the numerator of $CE'/E'A$ be multiplied by BD/DC it will become identical with the denominator of $BD'/D'C$. However, let us replace BD/DC by its equivalent $(BF/FA)\cdot(AE/EC)$ in the multiplication of the term $a^2((CE/EA)+1)\cdot((AF/FB)+1)$ occurring in the latter part of the numerator of $CE'/E'A$. By proceeding in this manner we may easily show that $(BD'/D'C)\cdot(CE'/E'A)\cdot(AF'/F'B)=1$. Thus Ceva's equation is satisfied for point P'.

If P be the centroid of triangle ABC, it is easily seen that $(BD/DC)=(CE/EA)=(AF/FB)=1$ and that

$$\frac{BD'}{D'C}=\frac{a^2-b^2+c^2}{a^2+b^2-c^2},\quad \frac{CE'}{E'A}=\frac{b^2-c^2+a^2}{b^2+c^2-a^2},\quad\text{and}\quad \frac{AF'}{F'B}=\frac{c^2-a^2+b^2}{c^2+a^2-b^2}.$$

It is not difficult to show that P' then becomes the orthocenter of triangle ABC and that the circle DEF is the nine point circle of the triangle. It is instructive to allow P to be the Gergonne point, incenter, circumcenter, symmedian point, either of the Brocard points, etc., of triangle ABC. The ratio values associated with P', the circle associate of point P, may then be computed. The use of Theorem 2 will also allow us to decide whether two points P and P' in the plane of triangle ABC have cevian triangles that are cyclic.

SUFFICIENT CONDITIONS FOR ENVELOPES IN n-SPACE

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We give here sufficient conditions for the existence of an envelope of a k-parameter family of $(n-1)$-dimensional surfaces in euclidean space R^n, $k \leqq n-1$. The special case $n=2$, $k=1$, is given in [1]. The criterion can be applied without attempting to find the envelope.

The familiar application is that of finding singular solutions of first order partial differential equations.

In the following, the index i ranges from 1 to n, j and h range from 1 to k, and r ranges from 1 to $k+1$. Double vertical bars indicate a matrix. Single vertical bars indicate a determinant. A subscript which is a variable indicates partial differentiation. Superscripts are indices, not powers.

Let

$$f(x, t) \equiv f(x_1, \cdots, x_n, t_1, \cdots, t_k) = 0 \tag{1}$$

be a family of loci in R^n, the space of $x_1, x_2, \cdots, x_n$, with f of class C' in a neighborhood of (x^o, t^o), $f(x^o, t^o)=0$, and $f_{x_i}(x^o, t^o)$ not all zero, so that each of the loci (1) is of dimension $n-1$. We say that the locus of the simultaneous equations

$$x_i = g_i(t_1, \cdots, t_k, s_1, \cdots, s_{n-k-1}) \tag{2}$$

with g_i of class C' neighboring (t^o, s^o), where $x_i^o = g_i(t^o, s^o)$, is an *envelope* of (1) if

$$\left\| \frac{D(g_1, \cdots, g_n)}{D(t_1, \cdots, t_k, s_1, \cdots, s_{n-k-1})} \right\| \tag{3}$$

is of rank $n-1$ (so that (2) is $(n-1)$-dimensional), and at each point (x^1) with $x_i^1 = g_i(t^1, s^1)$, for (t^1, s^1) in a neighborhood of (t^o, s^o), (2) is tangent to the surface (1) with $(t)=(t^1)$.

THEOREM. *Given $f(x_1, \cdots, x_n, t_1, \cdots, t_k)$, $2 \leqq n$, $1 \leqq k \leqq n-1$, with f of class C^2 in a neighborhood of (x^o, t^o), with $f, f_{t_1}, \cdots, f_{t_k}$ all zero at (x^o, t^o), with the matrix of $k+1$ rows and n columns*

$$\left| \frac{D(f, f_{t_1}, \cdots, f_{t_k})}{D(x_1, x_2, \cdots, x_n)} \right| \tag{4}$$

of rank $k+1$ at (x^o, t^o) and with

$$\left| \frac{D(f_{t_1}, \cdots, f_{t_k})}{D(t_1, \cdots, t_k)} \right| \neq 0 \text{ at } (x^o, t^o); \tag{5}$$

then the family (1) *of surfaces in R^n has an envelope* (2). *For fixed $t_1, \cdots, t_k$,* (2) *is tangent to* (1) *at each point of an $(n-k-1)$-spread in R^n (a point if $n-k-1=0$).*

Proof. From (4) we infer that the locus in (x, t) space, neighboring (x^o, t^o), of the simultaneous equations (1) and

$$f_{t_j}(x, t) = 0 \tag{6}$$

coincides with that of a system of equations

$$x_r = g_r(t_1, \cdots, t_k, x_{k+2}, \cdots, x_n) \equiv g_r(t, x_{k+2}, \cdots, x_n), \tag{7}$$

with g_r of class C', where, without loss of generality, we have chosen $x_1, x_2, \cdots, x_{k+1}$ as appearing in a nonzero minor of (4). Setting $g_m(t, x_{k+2}, \cdots, x_n) \equiv x_m$ for $m = k+2, \cdots, n$, we note that (7) has the same locus in (x, t) space as

$$x_i = g_i(t, x_{k+2}, \cdots, x_n),$$

which is in the form (2), with $s_1 = x_{k+2}, \cdots, s_{n-k-1} = x_n$.

By (5), simultaneous equations (6) are equivalent, in a neighborhood of (x^o, t^o) to a system

$$t_j = \tau_j(x_1, \cdots, x_n), \tag{8}$$

with τ_j of class C' neighboring (x^o) and $t_j^o = \tau_j(x^o)$. Hence (1) and (6) are equivalent, in a neighborhood of (x^o, t^o), to (8) and

$$h(x_1, \cdots, x_n) \equiv f[x_1, \cdots, x_n, \tau_1(x), \cdots, \tau_k(x)] = 0. \tag{9}$$

Since (6) and (8) are equivalent, we have $f_{t_j}[x, \tau(x)] \equiv 0$. Hence, from (9), by the chain rule,

$$\frac{\partial h}{\partial x_i} \equiv \frac{\partial}{\partial x_i}\{f[x, \tau_1(x), \cdots, \tau_k(x)]\} \equiv f_{x_i}[x, \tau_1(x), \cdots, \tau_k(x)] = 0.$$

Since, by (4), the f_{x_i} cannot all be zero, we therefore conclude that, at any (x) near (x^o) on (9), surface (9) is tangent to the member of (1) whose parameters are given by (8) with the given (x).

Now (9) and (7) give the same locus in (x)-space, since (1) and (6) are equivalent to (7) and are also equivalent to (8) and (9), while (8) imposes no restriction on the x's. Since $t_1, \cdots, t_k$ are independent for (7), we infer that (9) contains points on members of the family (1) for every (t) in a neighborhood of (t^o).

The locus, for fixed (t), in which the corresponding surface (1) has contact with the envelope (7) or (9) is evidently an $(n-k-1)$-spread in R^n, since $n-k-1$ of the x's are independent in (7).

Since (8) and (9) represent the same locus as (7) in (x, t)-space, all (x, t) which satisfy (7) also satisfy (8), and hence

$$t_j \equiv \tau_j[g_1(t, x_{k+2}, \cdots, x_n), \cdots, g_{k+1}(t, x_{k+2}, \cdots, x_n), x_{k+2}, \cdots, x_n].$$

Differentiating each side of the jth equation partially with respect to t_h, we obtain (using the Kronecker δ)

$$\delta_{jh} = \sum_r \frac{\partial \tau_j}{\partial x_r} \cdot \frac{\partial g_r}{\partial t_h} \cdot$$

Hence

$$\left\| \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} \right\| = \left\| \frac{D(\tau_1, \cdots, \tau_k)}{D(x_1, \cdots, x_{k+1})} \right\| \cdot \left\| \frac{D(g_1, \cdots, g_{k+1})}{D(t_1, \cdots, t_k)} \right\|,$$

and therefore

$$\left\| \frac{D(g_1, \cdots, g_{k+1})}{D(t_1, \cdots, t_k)} \right\| \tag{10}$$

is of rank k. Since $g_m(t, x_{k+2}, \cdots, x_m) \equiv x_m$ $(m=k+2, \cdots, n)$, we note that the matrix

$$\left\| \frac{D(g_1, \cdots, g_n)}{D(t_1, \cdots, t_k, x_{k+2}, \cdots, x_n)} \right\|$$

is of the form

$$\left\| \begin{matrix} \dfrac{D(g_1, \cdots, g_{k+1})}{D(t_1, \cdots, t_k)} & \dfrac{D(g_1, \cdots, g_{k+1})}{D(x_{k+2}, \cdots, x_n)} \\ 0 & \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} \end{matrix} \right\|, \tag{11}$$

where each zero signifies that all the indicated elements of the matrix are zero.

From (10) we infer that the rank of (11) is $k+(n-k-1)=n-1$, thus establishing (3) with $s_1=x_{k+2}, \cdots, s_{n-k-1}=x_n$, and the proof is complete.

We conclude with the following two observations:

1. Since familiar methods of proof show that any envelope must satisfy (1) and (6), we infer that (7), or (9), is the unique envelope of (1) for (x, t) near (x^o, t^o).

2. It might be surmised that if an envelope exists for (1), then if (1) is replaced by the family obtained from it by keeping *any* subset of the t's constant, the resulting family will have an envelope. That that is not true is shown by the following example. Consider the family

$$f \equiv x + sy + stz + 2s - t - 2 = 0$$

for (x, y, z, s, t) in a neighborhood of $(0, 0, 1, 1, -2)$. The hypotheses of the theorem are easily verified, and we infer that the family has an envelope. (The envelope can be given in the form $xz+y-2z+2=0$.) But if we take $t=-2$ and consider the resulting family

$$F \equiv x + sy - 2sz + 2s = 0,$$

and combine with $F_s \equiv y-2z+2=0$, we obtain $x=0$, $y-2z+2=0$, a line, and hence there is no envelope.

Reference

1. Angus E. Taylor, Advanced calculus, Ginn and Co., New York, 1955, p. 397.

A TRIPLE PRODUCT OF VECTORS IN FOUR-SPACE

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1. Introduction. The student in his study of three-dimensional vector analysis first becomes acquainted with the algebra of vectors. He becomes familiar with such operations as the addition, scalar multiplication, dot product, and cross multiplication of vectors and learns various geometrical interpretations for each of these operations.

Most of the operations of three-dimensional vector analysis are readily generalized to vector spaces of higher dimensions. For example, the dot product and the concept of orthogonality of three-vectors can be extended to vector spaces of n-dimensions in an obvious manner. For the cross product operation, however, no obvious extension to other vector spaces evidently exists. In this paper we propose to define a triple product, called a *ternary product*, of vectors in a space of four dimensions and to examine certain properties analogous to the properties of three-vector operations.

In the first portion of this paper we shall interpret the cross product in three-space in terms of a similar operation defined for vectors of two-space. Then, in an analogous manner, we shall interpret a ternary product of four-vectors in terms of the cross product of three-vectors.

2. Definitions and Terminology. (Many of the definitions and much of the terminology in this section may be found in [2].) We shall define a *vector* in n-dimensions as an ordered n-tuple with real components; e.g., $(a_1, a_2, \cdots, a_n)$. Such vectors will be denoted by capital letters $A, B, \cdots$. *Scalars*, denoted by lower case letters, $a, b, \cdots$, are real-valued functions and are, incidentally, the elements of the ordered n-tuples.

The set of n-vectors defined above is said to compose a *vector space* of dimension n, denoted by S_n, if the following two postulates hold:

(i) If X is a vector of S_n, and m is a scalar from the real number field, then mX is a uniquely determined vector of S_n. This external binary operation is called a *scalar multiple* of X, and is defined to be a vector whose elements are the products of the scalar m and the corresponding components of the vector X.

(ii) If X and Y are vectors of S_n, then $X+Y$ is a uniquely determined vector of S_n. This internal binary operation of *addition* of vectors X and Y is defined to be a vector whose elements are the sums of the corresponding components of X and Y. Thus,

$$X + Y = (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n). \tag{1}$$

From these definitions it follows that the vector $O = (0, 0, \cdots, 0)$ is the identity element of a set of vectors under addition. We also mention that n-vectors may be interpreted as points in n-dimensional space.

We now define some operations and properties of vectors belonging to S_n.

(a) Two vectors in S_n are said to be *linearly dependent* if and only if one is a scalar multiple of the other. If the two vectors are not linearly dependent, they are *linearly independent.*

(b) The *absolute value* of a vector X in S_n, called the *magnitude* of X and denoted by $|X|$, is defined to be the square root of the sum of the squares of the components of X. That is,

$$|X| = \sqrt{(x_1^2 + x_2^2 + \cdots + x_n^2)}. \tag{2}$$

This unitary operation associates a nonnegative real number with each vector in S_n.

(c) The *dot product* (or scalar product) of two vectors X and Y in S_n is defined to be a scalar that is the sum of the products of the corresponding components of X and Y, and is denoted as $X \cdot Y$; i.e.,

$$X \cdot Y = \sum_{i=1}^{n} x_i y_i, \tag{3}$$

where n is the number of components in the vectors X and Y. This binary operation associates a real number with each pair of vectors of S_n. It is readily shown that $(A+B)\cdot C = A \cdot C + B \cdot C$, which is the *distributive property*.

(d) Two vectors X and Y in S_n are said to be *orthogonal* if and only if

$$X \cdot Y = 0. \tag{4}$$

3. Vector Operations in S_3. In addition to the properties (*a*)–(*d*) defined in Section 2, in S_3 an interior binary operation, called the *cross product* operation, between vectors A and B, written $A \times B$, may be defined as the expansion of the following symbolic determinant:

$$A \times B = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1), \tag{5}$$

where i, j, and k are the elementary vectors (1, 0, 0), (0, 1, 0), and (0, 0, 1) respectively. It is readily shown that the cross product $A \times B$ yields a vector that is orthogonal to A and B. If we interpret vectors A and B as points in three-space with the zero vector O being the origin, then OA and OB can be interpreted as line segments. Thus the magnitude of $A \times B$ is a scalar that can be interpreted as the area of the parallelogram with the line segments from O to A and B as two of its sides. It is not difficult to show that $(A+B)\times C = A \times C + B \times C$, the *distributive property* of vectors in S_3.

Another operation of vectors in S_3 is the *triple scalar product*, which may be defined for three vectors A, B, and C as

$$A \cdot (B \times C) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}. \tag{6}$$

The triple scalar product may be interpreted geometrically as the volume of a parallelepiped with line segments from O to A, B, and C as three of the edges.

It is also shown in vector analysis in S_3 that the *triple vector product* $C \times (A \times B)$

may be written as

$$(7)\qquad C \times (A \times B) = (B\cdot C)A - (A\cdot C)B = \begin{vmatrix} A & B \\ A\cdot C & B\cdot C \end{vmatrix}.$$

Analogies for all of the definitions of this section will be considered later in the case of S_4.

4. Vector Operations in S_2. In the attempt to define an operation in S_2 analogous to the cross product of two vectors in S_3, we observe immediately that a cross product of two vectors in S_2 could not be defined to yield a vector mutually orthogonal to the two linearly independent vectors—the result would require a third dimension. We therefore define an operation on *one* vector in S_2 that will produce a vector orthogonal to the original vector.

Let the vector operation on a vector A in S_2 be denoted by A', called the *unitary vector operation*, and defined by the expansion of the following determinant:

$$(8)\qquad A' = \begin{vmatrix} i & j \\ a_1 & a_2 \end{vmatrix} = (a_2, -a_1),$$

where i and j are the elementary vectors (1, 0) and (0, 1) respectively. It follows from this definition that A' is orthogonal to A; i.e.,

$$(9)\qquad A\cdot A' = (a_1, a_2)\cdot(a_2, -a_1) = a_1a_2 - a_2a_1 = 0.$$

We next define an operation analogous to the triple scalar product (6) of vectors in S_3. Let the *double scalar product* of two vectors A and B in S_2 be defined as

$$(10)\qquad A\cdot(B') = (a_1, a_2)\cdot(b_2, -b_1) = a_1b_2 - a_2b_1 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$

Interpreting (10) geometrically, we find that $A\cdot(B')$ yields a scalar whose absolute value is equal to the *area* of a parallelogram with line segments from O to A and B as sides, while the triple scalar product in S_3 results in a scalar whose absolute value may be interpreted as the *volume* of a parallelepiped.

5. The Cross Product in S_3 with Respect to S_2. We now express the cross product of two vectors A and B in S_3, defined in (5), in the following manner:

$$(11)\qquad \begin{aligned} A \times B &= (a_2b_3 - a_3b_2)i + (a_3b_1 - a_1b_3)j + (a_1b_2 - a_2b_1)k \\ &= [(a_2, a_3)\cdot(b_3, -b_2)]i + [(a_3, a_1)\cdot(b_1, -b_3)]j + [(a_1, a_2)\cdot(b_2, -b_1)]k. \end{aligned}$$

Upon letting $A_1 = (a_2, a_3)$, $B_1 = (b_2, b_3)$, $A_2 = (a_3, a_1)$, $B_2 = (b_3, b_1)$, $A_3 = (a_1, a_2)$, and $B_3 = (b_1, b_2)$, we can write (11) as

$$(12)\qquad A \times B = A_1\cdot(B_1')i + A_2\cdot(B_2')j + A_3\cdot(B_3')k.$$

From (12) and the interpretation of (10) we see that the coefficient of i may be interpreted as the area of the *projection* of the parallelogram with sides OA and OB in S_3 on the yz-plane in an xyz-coordinate system. The coefficients of j

and k may be interpreted in a similar fashion as the areas of projection of the parallelogram in S_3 on the zx- and xy-planes respectively.

Hence, the magnitude of $A\times B$ may be interpreted geometrically as the square root of the sum of the squares of the areas in the zx-, xy-, and yz-planes which are projections of the parallelogram in S_3 with sides OA and OB.

6. A Triple Product of Vectors in S_4. In the preceding section we were able to express $A\times B$ in S_3 in terms of the double scalar product of vectors in S_2 spaces; i.e., the zx-, yz-, and xy-planes. This suggests that we may be able to express a vector operation in S_4 in terms of the triple scalar products of vectors in S_3 in an effort to interpret the S_4 vector operation geometrically.

We define a ternary interior operation in S_4, which we shall call the *ternary product*, of three vectors A, B, and C in S_4 by the expansion of the following determinant:

$$(13)\qquad A\otimes B\otimes C=\begin{vmatrix} i & j & k & h \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix},$$

a vector in S_4. The i, j, k, and h in (13) are the elementary vectors (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), and (0, 0, 0, 1). The properties which follow from this definition are analogous to those of the cross product of two vectors in S_3. Note that no associative property is involved in this definition.

From definition (13) and the properties of determinants, we note that

$$(14)\qquad (A\otimes B\otimes C)=(B\otimes C\otimes A)=-(C\otimes B\otimes A),\ \text{etc.}$$

That is, the sign of the ternary product is not affected by a cyclic permutation, but is changed by an acyclic permutation, of the vectors A, B, and C.

Upon considering the elementary vectors, we find that

$$(15)\qquad \begin{aligned} j\otimes k\otimes h&=i, & h\otimes k\otimes j&=-i,\\ i\otimes j\otimes h&=k, & h\otimes j\otimes i&=-k,\\ i\otimes h\otimes k&=j, & k\otimes h\otimes i&=-j,\\ k\otimes j\otimes i&=h, & i\otimes j\otimes k&=-h,\ \text{etc.}\end{aligned}$$

Note the similarity of these results with the cross product operation in S_3; i.e., $i\times j=k$, $k\times j=-i$, etc.

Analogous to the triple scalar product in S_3, we have that

$$(16)\qquad A\cdot(B\otimes C\otimes D)=\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}$$

which we shall call the *quadruple scalar product.*

From (16) it readily follows that

$$A\cdot(A \otimes B \otimes C) = B\cdot(A \otimes B \otimes C) = C\cdot(A \otimes B \otimes C) = 0; \tag{17}$$

that is, the ternary product $A \otimes B \otimes C$ yields a vector that is orthogonal to A, B, and C.

In the vector operations of S_3 we have the distributive property $(A+B)\times C = (A\times C)+(B\times C)$. A corresponding property results from definition (13) in S_4; i.e.,

$$(A + B) \otimes C \otimes D = (A \otimes C \otimes D) + (B \otimes C \otimes D). \tag{18}$$

Finally, recall that in S_3 we had the *triple vector product* of vectors A, B, and C expressed as

$$C \times (A \times B) = \begin{vmatrix} A & B \\ A\cdot C & B\cdot C \end{vmatrix} = A(B\cdot C) - B(A\cdot C),$$

a linear combination of vectors A and B, see (7). Geometrically the result may be interpreted as a vector lying in the plane of A and B. In an analogous manner in S_4 we now express the *quintuple vector product* of vectors A, B, C, D, and E as

$$D \otimes E \otimes (A \otimes B \otimes C) = \begin{vmatrix} A & B & C \\ A\cdot E & B\cdot E & C\cdot E \\ A\cdot D & B\cdot D & C\cdot D \end{vmatrix}, \tag{19}$$

a linear combination of vectors A, B, and C. This result can be interpreted as a vector lying in the space of vectors A, B, and C. It can be shown by expanding both sides of the equation in (19) that the quintuple vector product can be written as a determinant.

7. Geometric Interpretation of the Ternary Product of Vectors in S_4. Manning [1] describes four-space as a *hyperspace* composed of three-dimensional *hyperplanes*. For geometrical figures in S_3 that are bounded by plane surfaces, the corresponding figures in S_4 are bounded by three-dimensional *hypersurfaces*. We now proceed to justify the ternary product defined in (13).

In S_3 the cross product of two vectors, $A\times B$, may be interpreted as a vector whose magnitude is the area of a parallelogram with sides OA and OB, and the absolute value of the triple scalar product, $A\cdot(B\times C)$, may be interpreted as the volume of a parallelepiped with sides OA, OB, and OC. By analogy we might expect that the ternary product $A\otimes B\otimes C$ in S_4 would yield a vector whose absolute value would be the volume of a portion of a hypersurface with sides OA, OB, and OC, while the quadruple scalar product defined in (16) would yield a scalar whose absolute value is the volume of a *hypersolid* corresponding to the parallelepiped in S_3.

Upon writing the ternary product (13) as

$$\begin{aligned} A \otimes B \otimes C = {} & \{(a_2, a_3, a_4)\cdot[(b_2, b_3, b_4) \times (c_2, c_3, c_4)]\}i \\ & - \{(a_1, a_3, a_4)\cdot[(b_1, b_3, b_4) \times (c_1, c_3, c_4)]\}j \\ & + \{(a_1, a_2, a_4)\cdot[(b_1, b_2, b_4) \times (c_1, c_2, c_4)]\}k \\ & - \{(a_1, a_2, a_3)\cdot[(b_1, b_2, b_3) \times (c_1, c_2, c_3)]\}h, \end{aligned} \tag{20}$$

we see that the coefficients of i, j, k, and h may be interpreted as triple scalar products of vectors in four hyperplanes or three-spaces. For example, the coefficient of i in (20) may be interpreted as the magnitude of the volume of a portion of a hypersurface in the jkh-hyperplane. The coefficients of j, k, and h can be interpreted in a similar manner. Since hypersolids are bounded by three-dimensional hypersurfaces, it is logical to expect the projection of a hypersurface in a hyperplane to be three-dimensional. This would explain why the ternary product in S_4 should involve three vectors instead of two, as is the case in S_3.

8. Conclusion. Our reasoning in the latter portion of this paper is based almost entirely on inferences from the first part. That is, the geometrical interpretation of the ternary product in S_4 in terms of hyperplanes has been analogous to the geometrical interpretation of the cross product in S_3 in terms of two-spaces. Our definition of the ternary product in S_4, however, seems to be justifiable in terms of four-dimensional geometry. Although our study of S_4 has been somewhat limited, it seems to indicate that a vector analysis of four dimensions is not beyond reason.

Prepared in a National Science Foundation Undergraduate Science Education Program in Mathematics at Colorado State University under the direction of Professor Stein.

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2. F. Max Stein, An introduction to vector analysis, Harper, New York, 1963.

BAYES' FORMULA AND A PRIORI PROBABILITIES IN THE GAME OF BRIDGE

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There has been some controversy between bridge experts and mathematicians regarding the use of *a priori* probabilities in computing probabilities in the middle of a hand, i.e., after some cards have been played. The mathematicians have argued that the correct approach is the one which makes use of a slight variation of Bayes' formula [1] and [5]. However certain variables in this formula are not easily ascertained and even its supporters [5] admit that: "The main difficulty with the analysis is that of making realistic estimates . . . . But a good bridge player can . . . . "

The bridge players on the other hand have continued to make use of the *a priori* probabilities even though they have been told that these do not always give the best estimates. Some have attempted to use these *a priori* probabilities with no adjustments whatsoever [4] even though a good deal of extra information has been obtained by the play of the cards. Others have tried to remove all cases known to be impossible and then considered the *a priori* probabilities adjusted [3]. Though both of these approaches give wrong answers most of the we see that the coefficients of i, j, k, and h may be interpreted as triple scalar products of vectors in four hyperplanes or three-spaces. For example, the coefficient of i in (20) may be interpreted as the magnitude of the volume of a portion of a hypersurface in the jkh-hyperplane. The coefficients of j, k, and h can be interpreted in a similar manner. Since hypersolids are bounded by three-dimensional hypersurfaces, it is logical to expect the projection of a hypersurface in a hyperplane to be three-dimensional. This would explain why the ternary product in S_4 should involve three vectors instead of two, as is the case in S_3.

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The bridge players on the other hand have continued to make use of the *a priori* probabilities even though they have been told that these do not always give the best estimates. Some have attempted to use these *a priori* probabilities with no adjustments whatsoever [4] even though a good deal of extra information has been obtained by the play of the cards. Others have tried to remove all cases known to be impossible and then considered the *a priori* probabilities adjusted [3]. Though both of these approaches give wrong answers most of the time, there is some mathematical justification (as we shall see) for this apparent inconsistency of bridge players.

A bridge player makes use of two quite different techniques. The first and most important is his bridge judgment. By listening carefully to the bidding, both to what is bid and what is not bid, and by considering the tone of voice, the speed used in making bids, he makes guesses as to the distribution of the opponents' cards and to the placing of certain key cards. After the bidding is over and the play begins, he observes what is played, what is not played, the manner in which the cards are played, and every little nuance of movement, tension, or nervousness. On the basis of this often subtle information he makes decisions, and quite often he will get a "feeling" that West rather than East, has a certain card. There is nothing scientific in this and often the most successful player will not be able to explain why he took a certain view. It may have been based on "electricity" of the moment. Occasionally it is based on an intimate knowledge of the habits of his opponents.

The second technique used is the purely scientific one of mathematical probability. And of course the bridge master uses both techniques as best he can. Against extremely good opponents one can get only little information from their mannerisms and one is thus forced to rely more heavily on probability.

Thus the bridge master often turns to pure probability when he faces unknown opponents or when he does not have a "feel" for the situation. It is here that Bayes' formula does not appeal to him, for to use it successfully one must be able to make shrewd estimates of certain variables. To oversimplify the situation we may say that when a bridge player cannot make realistic estimates, and turns to probability theory he does not want to be told that he must use a formula which works only after he makes realistic estimates!

Then what shall one use for the unknown variables in Bayes' formula? And perhaps just as awkward, how can one realistically carry out the often laborious computation necessary to evaluate Bayes' formula, in actual play at the table? In [1], page 388, the technique of equating equivalent cards is suggested but this does not seem to go far enough.

In [2] we suggested that one use the *a priori* probabilities, removing distributionally impossible cases, but not removing all impossible cases. For example if East-West have four spades between them, let us say the queen, jack, six, five of spades, and if a round of spades is played with East playing the jack and West playing the five, then we know that these four spades were not divided 4-0 or 0-4. This is then a distributionally impossible case. We may then restrict our considerations to the possible distributions, namely, 2-2, 3-1 or 1-3. However since West played the five, it is in fact impossible for East to have had the five in this situation. Thus we know more than the simple fact that the spades were not 4-0 or 0-4. But we shall ignore this extra information.

We propose to prove that for two important cases, this approach gives a sound estimate of Bayes' formula, and avoids the laborious computation. This also makes clear the bridge players' reason for persistently using these condemned *a priori* probabilities.

In spite of the many practical difficulties in using Bayes' formula there are important cases where it can be used most effectively. Thus the aspirant to bridge mastery should acquaint himself with Bayes' formula and remember some of its fundamental results just as he remembers the *a priori* probabilities.

Problem A. We assume that the declarer, South, knows how a certain suit, say spades, is distributed among his opponents. Suppose then South knows that West started with r spades and East started with s spades. These $r+s$ spades consist of 1 important or high spade and $r+s-1$ unimportant or small spades. The so-called *a priori* probability that West has the high spade is clearly $r/(r+s)$. (The use of the term '*a priori*' here is somewhat misleading for it is used at the point where South has discovered the distribution of the spade suit. Generally he would not know this until several cards have been played and thus this point is not at the beginning of the play. It is, however, commonly used in this way in this context.) Then k rounds of spades are played, $k<r$, $k<s$, and both East and West play k little spades each. Now what is the probability that West has the high spade?

The answer is theoretically quite simple. We consider only those original cases that are still possible, call them $c_1, c_2, \cdots, c_n$. These c_i all have the same *a priori* probabilities.

To each of these cases we associate

$P_i=$ the probability that given case c_i, East and West will each play their k little spades in the manner they have played them.

Let $c_1, c_2, \cdots, c_t$ be those cases in which West holds the high spade, and let $c_{t+1}, \cdots, c_n$ be those cases in which East holds the high spade. Then, by Bayes' formula, the probability that West holds the high spade is

$$\frac{P_1 + P_2 + \cdots + P_t}{P_1 + P_2 + \cdots + P_t + P_{t+1} + \cdots + P_n}.$$

The trouble is that we do not know the values of the P_i. They depend on the strategy or the whim of East and West. These are the variables that the good bridge player must make realistic estimates of. Faced with unknown opponents this is most difficult.

The best strategy for East and West is to play their little spades at random. This then is a reasonable approach and in that case $P_1=P_2=\cdots=P_t$ while $P_{t+1}=P_{t+2}=\cdots=P_n$. But then we have:

THEOREM 1. *If East and West play their little spades at random, then the answer to Problem A is $r/(r+s)$, the a priori probability.*

Proof. There are

$$\binom{r+s-2k-1}{r-k-1}$$

cases left in which West holds the high spade. To each of these we assign

$$\frac{1}{(r-1)(r-2)\cdots(r-k)s(s-1)\cdots(s-k+1)},$$

the probability that West and East will have each played the k little cards they did play, from the given case.

There are

$$\binom{r+s-2k-1}{s-k-1}$$

cases left in which West does not have the high spade and to each of these we assign

$$\frac{1}{(s-1)(s-2)\cdots(s-k)r(r-1)\cdots(r-k+1)}.$$

Then by Bayes' Theorem, the probability that West has the important card is

$$\frac{\dfrac{\binom{r+s-2k-1}{r-k-1}}{(r-1)r-2)\cdots(r-k)s(s-1)\cdots(s-k+1)}}{\dfrac{\binom{r+s-2k-1}{r-k-1}}{(r-1)r-2)\cdots(r-k)s(s-1)\cdots(s-k+1)}+\dfrac{\binom{r+s-2k-1}{s-k-1}}{(s-1)(s-2)\cdots(s-k)r(r-1)\cdots(r-k+1)}}$$

and this simplifies down to $r/(r+s)$.

Problem B. East and West hold r spades between them, $r<13$. Suppose that k rounds of spades are played, $k\leqq(r-1)/2$, and that East and West each play a spade on every round. Then there is at least one spade still left with East, West and of course both East and West began with at least k spades each. What is the probability that West began with s spades and East began with $r-s$ spades, $k\leqq s$, $k\leqq r-s$?

Here again the answer is theoretically straightforward but again we run into the practical difficulty of estimating the P_i. To make progress we can again assume that East and West play their spades at random but here the assumption is not as sound as in Problem A. In Problem A, with only one high spade out, it is quite reasonable to assume that the small spades will be played at random but here in Problem B, there may be several high spades with East, West and though they may play their small cards at random, they will not play their high cards at random. Nonetheless this assumption of randomness is perhaps the best practical approach. We therefore use it and begin a listing of the possible cases. We note that East, West began with $26-r$ cards that were not spades.

There are $\binom{26-r}{13-k}$ cases in which West had precisely k spades and to these we associate $\{k(k-1)\cdots 3\cdot 2\cdot 1\cdot(r-k)(r-k-1)\cdots(r-2k+1)\}^{-1}$. There are $\binom{26-r}{12-k}\cdot\binom{r-2k}{1}$ cases in which West had precisely $k+1$ spades and to these we associate $\{(k+1)k(k-1)\cdots 3\cdot 2\cdot(r-k-1)(r-k-2)\cdots(r-2k)\}^{-1}$. Continuing, we find there are $\binom{26-r}{13-s}\binom{r-2k}{s-k}$ cases in which West had precisely s spades and to these we associate $\{s(s-1)\cdots(s-k+1)(r-s)(r-s-1)\cdots(r-s-k+1)\}^{-1}$. And finally there are $\binom{26-r}{13-r+k}\binom{r-2k}{r-2k}$ cases in which West had precisely $r-k$ spades and to these we associate $\{(r-k)(r-k-1)\cdots(r-2k+1)\cdot k(k-1)\cdots 1\}^{-1}$.

Then by Bayes' formula, the probability that West had s spades is:

$$\frac{\binom{26-r}{13-s}\binom{r-2k}{s-k}\{s(s-1)\cdots(s-k+1)\cdot(r-s)(r-s-1)\cdots(r-s-k+1)\}^{-1}}{\binom{26-r}{13-k}\{k(k-1)\cdots 1\cdot(r-k)\cdots(r-2k+1)\}^{-1}+\cdots+\binom{26-r}{13-i}\binom{r-2k}{i-k}\{i(i-1)\cdots(i-k+1)\cdot(r-i)\cdots(r-i-k+1)\}^{-1}+\cdots+\binom{26-r}{13-r+k}\binom{r-2k}{r-2k}\cdot\{(r-k)\cdots(r-2k+1)\cdot k\cdots 1\}^{-1}}$$

$$=\binom{26-r}{13-s}\left\{\frac{\dfrac{(r-2k)!}{(s-k)!(r-s-k)!s(s-1)\cdots(s-k+1)\cdot(r-s)(r-s-1)\cdots(r-s-k+1)}}{\sum_{i=k}^{i=r-k}\binom{26-r}{13-i}\binom{r-2k}{i-k}\{i(i-1)\cdots(i-k+1)(r-i)(r-i-1)\cdots(r-i-k+1)\}^{-1}}\right\}.$$

This simplifies down to:

$$\binom{26-r}{13-s}\left\{\frac{\dfrac{1}{(r-s)!s!}}{(26-r)!\sum_{i=k}^{i=r-k}\dfrac{1}{(13-i)!(13-r+i)!i!(r-i)!}}\right\}.$$

Now let us consider the *a priori* probabilities regarding the distribution of the spade suit. The probability (*a priori*) that West had i spades is

$$\frac{\binom{r}{i}\binom{26-r}{13-i}}{\binom{26}{13}},\qquad i=0,1,\cdots,r.$$

Let us remove those cases that we know are distributionally impossible, i.e., where West had $0, 1, 2, \cdots, k-1$ or $r, r-1, \cdots, r-k+1$ spades. Then consider all the remaining cases, even though some of these are in fact impossible if we take into account the actual cards played during the k rounds of spades. Thus we are going to ignore the information about the actual cards played but we are going to use the information about which distributions are impossible.

Then the probability that West had s spades is:

$$\frac{\binom{r}{s}\binom{26-r}{13-s}}{\binom{26}{13}}\Bigg/\sum_{i=k}^{i=r-k}\frac{\binom{r}{i}\binom{26-r}{13-i}}{\binom{26}{13}}$$

$$= \binom{26-r}{13-s}\frac{r!}{(r-s)!s!} \Big/ \sum_{i=k}^{i=r-k} \frac{r!}{i!(r-i)!}\,\frac{(26-r)!}{(13-i)!(13-r+i)!}$$

$$= \binom{26-r}{13-s}\frac{1}{(r-s)!s!} \Big/ (26-r)! \sum_{i=k}^{i=r-k} \frac{1}{i!(r-i)!(13-i)!(13-r+1)!}\,.$$

This is precisely the same answer as the one given by Bayes' formula under the assumption that East and West played their spades at random. We thus have

THEOREM 2. *If East and West play their spades at random, then the answer to Problem B can be obtained by removing the distributionally impossible cases and then using the a priori probabilities.*

Since bridge players often know, or memorize, the *a priori* probabilities of suit distribution, the mental arithmetic involved in using Theorem 2 is quite easy and therefore this theorem can be put to good practical use.

However it is only fair to close with a situation in which Bayes' theorem can be used effectively though here again we must estimate the elusive P_i's. The problems we consider are a mixture of Problems A and B. We seek knowledge about both the distribution of the suit and the location of a missing high card.

Problem C_1. Declarer holds A1098 and his dummy has KJ765 in spades. The A is played and West plays the 2 while East plays the 3. (Dummy plays the 5.) Now the 10 is led and West plays the 4. What is the probability that West has the Q?

There are only two possible patterns: West had 4, 2 or West had Q, 4, 2. In the pattern when West had the 4, 2, assuming he played them at random (as he should) the probability that he would play the 2 first and then the 4 is 1/2. East, holding Q, 3, would always play the 3. Thus the probability associated with this pattern is 1/2.

For the pattern where West had Q, 4, 2, we again associate 1/2 for he would never play the Q on the first two rounds.

Now how many cases are there? Since East, West have 4 spades and 22 nonspades between, the number of cases in which West has the 4, 2 of spades is $\binom{22}{11}$ while the number of cases in which West has the Q, 4, 2 of spades is $\binom{22}{10}$. To each of these cases we associate the probability 1/2. Then by Bayes' formula the probability that West had Q, 4, 2 of spades is

$$\frac{\binom{22}{10}\frac{1}{2}}{\binom{22}{10}\frac{1}{2}+\binom{22}{11}\frac{1}{2}} = \frac{\dfrac{22!}{10!12!}}{\dfrac{22!}{10!12!}+\dfrac{22!}{11!11!}} = \frac{\dfrac{1}{10!12!}}{\dfrac{11+12}{11!12!}} = \frac{11}{23}\,.$$

Consequently the probability that East had the Q is $1-11/23=12/23$ and therefore playing the K is slightly better than finessing. This fits in with the simpler view that West has 11 unknown cards while East has 12 unknown cards and therefore it is 12/23 that East has the Q.

Now let us consider

Problem C_2. [1] Declarer holds A1098, dummy has K7654 in spades. The A is led, West plays the 3 and East plays the J. Now the 10 is led and West plays the 2. What is the probability that West has the Q?

Again there are only two possible patterns: $\binom{22}{11}$ cases where West had 3, 2 of spades and $\binom{22}{10}$ cases where West had Q, 3, 2. But what probabilities should we attach to these cases? If West holds 3, 2, he plays the 3 first and then the 2, half the time (assuming as usual that he plays these small cards at random). East, however, holding Q, J will play the J first also half the time and therefore we must associate $1/2\cdot 1/2=1/4$ to these $\binom{22}{11}$ cases. In the cases where West had Q, 3, 2, he again would play the 3 first and the 2 second half the time. East, however, holding the lone J will always play it on the first round and therefore the probability we must associate with these $\binom{22}{10}$ cases is 1/2. Thus the probability that West holds the Q is

$$\frac{\binom{22}{10}\frac{1}{2}}{\binom{22}{10}\frac{1}{2}+\binom{22}{11}\frac{1}{4}}=\frac{11}{11+6}=\frac{11}{17}.$$

In this case, then, the finesse is a much better play than the K.

Such surprising results in cases that seem so similar are typical of Bayes' formula and omitting it from one's arsenal is a distinct error.

Note that in Problem C_1 or C_2, any attempt to use the *a priori* probabilities, even excluding the impossible distributions, fails. For the two possible distributions have *a priori* probabilities of 40.7 percent for 2-2 and 24.9 percent for 3-1 and thus the conclusion would be that the drop has a 62 percent chance whereas it has only a 52 percent chance in Problem C_1 and only a 35 percent chance in Problem C_2.

In the case when the P_i are unknown and no simpler manipulation of *a priori* probabilities helps, the proper mathematical approach is the Game Theory view. One should consider all possible East-West strategies, and the P_i associated with each strategy. Then one can obtain the usual game theory matrix and obtain an optimal strategy for the declarer. However, this is quite impractical at the table and leads to an answer which in general will be a mixed strategy, one that doesn't seem to suit the bridge player.

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TRIANGULATION OF A TRIANGLE

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The following geometrical problem arises in the theoretical treatment of 3-component alloys:

Given a triangle ABC (the "base" triangle on which the composition of the alloy in terms of constituents ABC is represented), given further three more series of points as follows:

$$A_1, A_2, \cdots, A_\alpha \text{ on the side } BC$$
$$B_1, B_2, \cdots, B_\beta \text{ on the side } AC$$
$$C_1, C_2, \cdots, C_\gamma \text{ on the side } AB.$$

(These points evidently represent binary compounds of various compositions.)

Now draw a series of straight lines between the points mentioned, dividing thus ABC in a number of sub-triangles subject to the following conditions:

(a) These dividing lines must not intersect inside the triangle ABC.

(b) Each of the points mentioned must appear as a corner-point of at least one triangle.

The question to be answered is then: In how many different ways is it possible to carry out a "triangulation"?

The problem is in each particular case evidently determined by the numbers α, β, γ, of "binary" points present and, for short, we denote the wanted number as $N(\alpha, \beta, \gamma)$. The following considerations show how N can be calculated starting from the simpler kind of problem of the type: $N(\alpha, 0, 0)$ proceeding to the intermediate problem of the type $N(\alpha, \beta, 0)$ and concluding with the general problem of type $N(\alpha, \beta, \gamma)$.

A. The problem of type $N(\alpha, 0, 0)$. This is trivial. From inspection of Figure 1 it follows that there is only one way in which the required triangulation can be carried out; namely by connecting

(1) $$A \text{ to } A_1, A_2, \cdots, A_\alpha, \quad \text{thus} \quad N(\alpha, 0, 0) = 1.$$

B. The problem of type $N(\alpha, \beta, 0)$. From inspection of Figure 2 it follows that starting at corner C the first line to be drawn must evidently be A_1 to B_1. Every line further away from C would leave either A_1 or B_1 unconnected. We visualise the line just drawn as well as subsequent dividing lines as a series of cuts produced by a knife, which "walks" towards the line AB. Each cutting position of this knife arises from the preceding one either by pushing the right or the left side of the knife forward to the next point on CA or CB. Thus a particular mode of cutting up the base triangle is mirrored in a cutting instruction consisting of a sequence of symbols l or r. The end position of the knife is reached when it coincides with AB. Only when it has reached this position has it produced out of the infinite angular sheet ACB all the components of the jigsaw that makes up the triangle ABC. An alternative solution in which the cutting up process is applied to the triangle ABC instead of the infinite angle ACB is given in Appendix 1. The calculation of $N(\alpha, \beta, 0)$ is thus reduced to the calculation of the possible different number of sequences $(l \cdots r \cdots r \cdots l)$ that can be written down for such a walk.

Now, each cutting instruction must contain a total of α symbols l and β symbols r. The number of different sequences is thus

$$N(\alpha, \beta, 0) = \frac{(\alpha + \beta)!}{\alpha!\beta!}. \tag{2}$$

(If all the symbols l and r were tagged to be different, there would be $(\alpha+\beta)!$ possible sequences. We have to divide by $\alpha!$ and $\beta!$ when we abolish those "tags.")

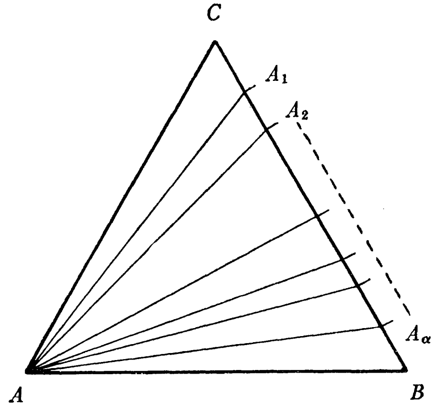

FIG. 1. Problem $N(\alpha, 0, 0)$.

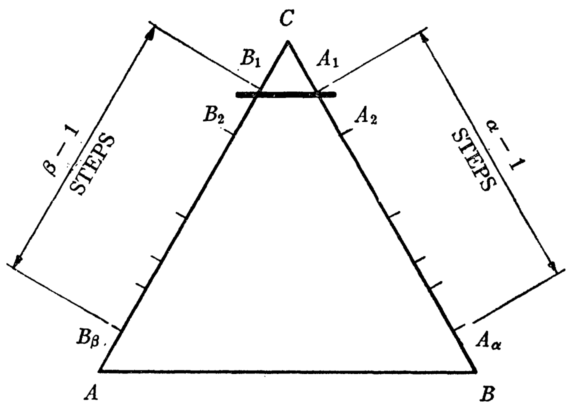

FIG. 2. Problem $N(\alpha, \beta, 0)$.

C. The problem of type $N(\alpha, \beta, \gamma)$. Figure 3 illustrates this case.

When one carries out a series of triangulations one becomes aware of the fact that there are two distinct possibilities. In the first case one of the corner-points A, B, or C is met by one or more cutting lines. In the second case none of the points A, B, or C is met by a cutting line.

(a) In the first case it will be noted that only one of the points A, B, or C can take part in a particular triangulation. (If C is this point, then there must be at least one cutting line from C to one of the points $C_1, \cdots, C_\gamma$; the existence of such a line prohibits any cutting line from B to $B_1, \cdots, B_\beta$ or from A to $A_1, \cdots, A_\alpha$.)

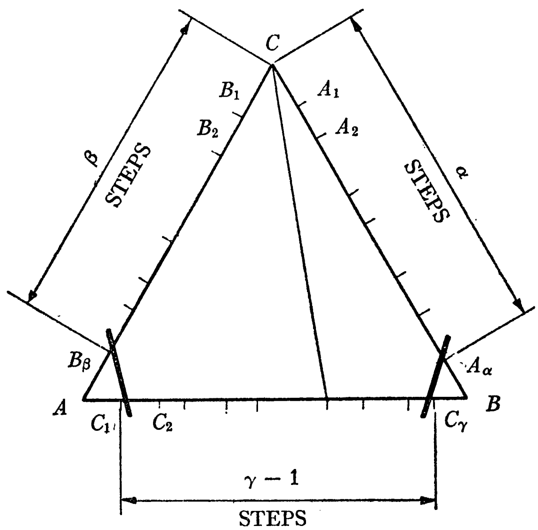

FIG. 3. Problem $N(\alpha, \beta, \gamma)$. Calculation of partial sum N_C.

Let us denote by N_A, N_B or N_C the number of different triangulations which include either A, B, or C on a cutting line. Figure 3 illustrates the calculation of N_C. The first position of the cutting line passes through C_1 and B_β. The final position of the cutting line is, C_γ and A_α. Intermediate positions are reached by a sequence of l and r operations. Each complete walk of the cutting knife contains here $(\alpha+\beta)$ instructions l and $(\gamma-1)$ instructions r. Thus the number of possibilities is

$$N_C = \frac{(\alpha+\beta+\gamma-1)!}{(\alpha+\beta)!(\gamma-1)!}. \tag{3}$$

By cyclic permutation we get

$$N_A = \frac{(\alpha+\beta+\gamma-1)!}{(\beta+\gamma)!(\alpha-1)!}, \tag{4}$$

$$N_B = \frac{(\alpha+\beta+\gamma-1)!}{(\gamma+\alpha)!(\beta-1)!}. \tag{5}$$

(b) The calculation of the second case is illustrated by Figure 4. The numbering in Figure 4 has been changed to allow cyclic permutation.

There are three cutting lines, closest to the corners A, B, C of which we know already the positions; they are the lines A_1B_β, B_1C_γ, C_1A_α. These three lines intersect somewhere outside the triangle. If we walk these lines "forward" (towards the centre of the triangle) their points of intersection come closer to the sides of the triangle until in the end they intersect just at the sides of the triangle. If this position is reached we say the cutting lines have "met"; they cannot walk further without infringing the condition that the cutting lines must not intersect inside the base triangle ABC. In their meeting position the three cutting lines define a triangle (crosshatched in Figure 4) the "irreducible triangle." Now it is seen that according to the "walking instructions" for the cutting lines we get a whole series of different irreducible triangles and each such triangle is associated with a number of different possibilities of triangulation, namely the different possibilities for the three trapezoidal areas that have been cut off between the starting and the finishing position of each cutting line.

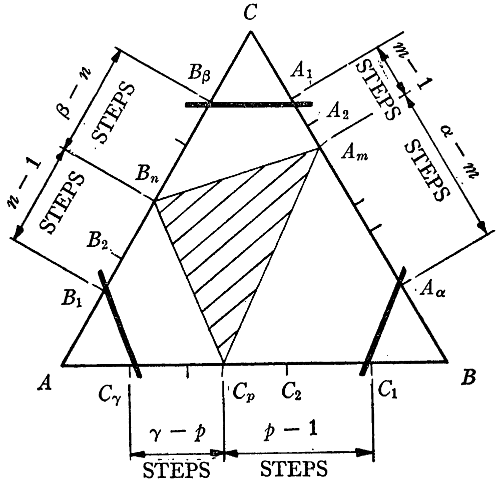

FIG. 4. Problem $N(\alpha, \beta, \gamma)$. Calculation of partial sum N_{ABC}.

Let us find out how many different irreducible triangles can be formed. The corners of such a triangle must be formed by a triplet $A_mB_nC_p$ of points selected from $A_1, \cdots, A_\alpha$, $B_1, \cdots, B_\beta$, and $C_1, \cdots, C_\gamma$. This gives altogether $\alpha\cdot\beta\cdot\gamma$ different ways of selection. The triplet of parameters m, n, p defines also the possible "walking" instructions for the triangulation of the trapezoidal areas. For instance the area close to A is cut up by $(n-1)$ instruction l and $(\gamma-p)$ instruction r, so that the number of possibilities is

$$\Delta_{np} = \frac{(n - 1 + \gamma - p)!}{(n - 1)!(\gamma - p)!}.$$

Similarly

$$\Delta_{pm} = \frac{(p - 1 + \alpha - m)!}{(p - 1)!(\alpha - m)!} \quad \text{and} \quad \Delta_{mn} = \frac{(m - 1 + \beta - n)!}{(m - 1)!(\beta - n)!}.$$

Altogether associated with the irreducible triangle $A_mB_nC_p$ there are then $\triangle_{np}\cdot\triangle_{pm}\cdot\triangle_{mn}$ different ways of triangulation. The overall result for this second mode of triangulation is thus

$$N_{ABC} = \sum_{m,n,p} \triangle_{np}\triangle_{pm}\triangle_{mn} \tag{6}$$

where $1 \leqq m \leqq \alpha$, $1 \leqq n \leqq \beta$, $1 \leqq p \leqq \gamma$, and the total result

$$N = N_A + N_B + N_C + N_{ABC}. \tag{7}$$

It is possible to replace the triple sum of equation (6) by a double sum if one proceeds as follows:

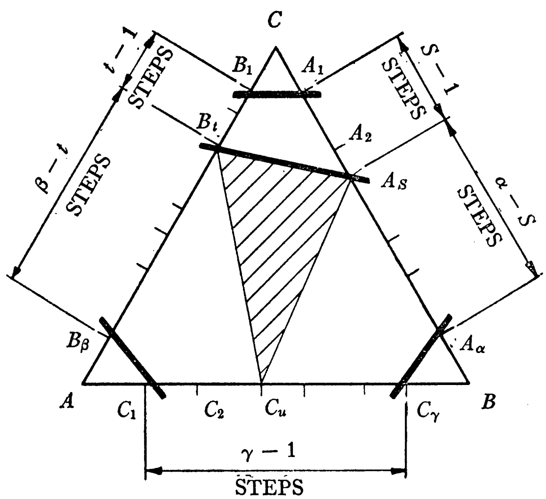

FIG. 5. Problem $N(\alpha, \beta, \gamma)$. Alternative calculation of partial sum N_{ABC}.

In Figure 5 we have renumbered the points along AC and AB and plotted a cutting line from A_s to B_t. There are evidently $\alpha\cdot\beta$ different cutting lines that can be drawn in this manner. Each of these lines divides the original triangle into a triangle near C (triangle A_sB_tC) and into a trapezoidal area AB_tA_sB.

The top triangle can be triangulated by a cutting line starting from position A_1B_1 walking towards A_sB_t, as the result of $(s-1)$ instructions l and $(t-1)$ instructions r giving

$$N_\triangle = \frac{(s+t-2)!}{(s-1)!(t-1)!}$$

possibilities of triangulation.

Each trapezoidal area can be triangulated by a cutting line walking from position C_1B_β to say, a position C_uB_t, swinging round to a position C_uA_s as the result of an instruction l and continuing to an end position $C_\gamma A_\alpha$.

The triangle $C_uB_tA_s$ is evidently the irreducible triangle associated with this particular walk. There are $\beta-t$ instructions l for the walk along AC and $\alpha-s$ instructions l for the walk along CB, altogether thus $\alpha+\beta-s-t+1$ instructions l and $\gamma-1$ instructions r. The total number of possibilities of triangulation for the trapezoidal area is thus

$$N_\square = \frac{(\alpha+\beta+\gamma-s-t)!}{(\alpha+\beta-s-t+1)!(\gamma-1)!}.$$

The total number of triangulations is then

$$N_{ABC} = \sum_{s,t} N_\triangle \cdot N_\square$$

as the triangulations $N_\triangle$ and $N_\square$ are completely independent of each other. (Advancing the boundary of the triangular area by one step l or r leads always to a new set of triangulations, characterised by a new set of irreducible triangles.

We can thus write

$$N_{ABC} = \sum_{s,t} \frac{(s+t-2)!}{(t-1)!(s-1)!} \frac{(\alpha+\beta+\gamma-s-t)!}{(\alpha+\beta-s-t+1)!(\gamma-1)!}$$

where $1<s<\alpha$ and $1<t<\beta$.

Alternative derivation of $N(\alpha, \beta, 0)$. If one applies the cutting process of Section B to a triangle CAB instead of the infinite sheet defined by the angle CAB then the number of cutting instructions is reduced in each case by 1. The cutting line finishes then in position AA_α or $B_\beta B$. The total number of possibilities of the former kind are

$$\frac{(\beta+\alpha-1)!}{\beta!(\alpha-1)!}.$$

The total number of possibilities of the second kind are

$$\frac{(\beta-1+\alpha)!}{(\beta-1)!\alpha!}.$$

Their sum is

$$N(\alpha, \beta, 0) = (\alpha+\beta-1)!\left[\frac{1}{(\alpha-1)!\beta!} + \frac{1}{\alpha!(\beta-1)!}\right]$$

$$= \frac{(\alpha+\beta-1)!}{(\alpha-1)!(\beta-1)!}\left[\frac{1}{\alpha}+\frac{1}{\beta}\right] = \frac{(\alpha+\beta)!}{\alpha!\beta!}.$$

Acknowledgement. The author is indebted to Mr. A. Prince of the Hirst Research Centre who acquainted him with the problem and who materially assisted him in testing the solution arrived at.

RANDOM WALK WITH TRANSITION PROBABILITIES THAT DEPEND ON DIRECTION OF MOTION

LEON COHEN, Yale University

1. Introduction. In the classical random walk [1], with absorbing barriers at 0 and a, the transition probabilities p and q are defined as the probability that the particle will go from k to $k+1$ or $k-1$ respectively, for $0<k<a$. We consider here a random walk with transition probabilities that depend on the direction in which the particle is traveling.

Let t be the probability that a particle, executing a random walk on $(0, a)$, will go from position k to $k+1(k-1)$ if in the previous step it was in position $k-1(k+1)$. Similarly, let $r(r+t=1)$ be the probability that the particle goes from k to $k+1(k-1)$ if in the previous step it was in position $k+1(k-1)$. That is to say that the particle changes direction or does not change direction with probabilities r and t respectively, whether it is going from right to left or left to right. If the particle does change direction we then say that it was reflected and if it does not we say it is transmitted.

2. Probability of absorption. Let L_z be the probability that the particle, starting at z and moving toward $z-1$ in the first step, is absorbed at 0. Also let R_z be the probability of absorption at 0 if the particle is initially moving to the right, that is from z to $z+1$.

Consider the case for R_z. After the first step the particle will be at $z+1$ and traveling either right or left with probabilities t and r respectively. If it is transmitted then the probability of absorption at 0 is R_{z+1} and L_{z+1} if it is reflected. We thus have

$$R_z = tR_{z+1} + rL_{z+1} \tag{1}$$

for $0<r<1$ and $z=1, 2, \cdots, a-1$. Similar considerations for L_z lead to

$$L_z = rR_{z-1} + tL_{z-1} \tag{2}$$

for $0<r<1$ and $z=2, 3, \cdots, a-1$. We define the boundary conditions as

$$L_1 = 1, \qquad R_{a-1} = 0. \tag{3}$$

(1) and (2) are two simultaneous difference equations which may be easily solved. From (1) we have

$$L_z = \frac{1}{r}(R_{z-1} - tR_z) \tag{4}$$

for $z=2, 3, \cdots, a-1$, and

$$L_{z-1} = \frac{1}{r}(R_{z-2} - tR_{z-1}) \tag{5}$$

for $z=3, 4, \cdots, a-1$. Substituting (4) and (5) in (2) we have

$$R_z - 2R_{z-1} + R_{z-2} = 0. \tag{6}$$

The solution of (6) is immediate:

$$R_z = C + Dz, \tag{7}$$

where C and D are arbitrary constants. Substituting (7) in (1) we also have

$$L_z = C + D\left(z - \frac{1}{r}\right). \tag{8}$$

Using the boundary conditions on (7) and (8) we find that the constants turn out to be:

$$C = \frac{(a-1)r}{ar - 2r + 1}, \qquad D = -\frac{r}{ar - 2r + 1}.$$

Therefore

$$R_z = \frac{r(a - z - 1)}{ar - 2r + 1}, \tag{9}$$

and

$$L_z = \frac{r(a - z) + t}{ar - 2r + 1} = R_z + \frac{1}{ar - 2r + 1}. \tag{10}$$

That (9) and (10) satisfy (1) and (2) for $z=1, 2$, can be checked by direct substitution.

3. Expected duration. For a particle that begins at z, let N_z be the expected duration of the walk if the particles first step is to $z+1$, and M_z if the initial step is to $z-1$. If the particle is at $z+1$, say, then the expected duration is M_{z+1} or N_{z+1} depending on whether it was reflected or transmitted. We therefore have

$$N_z = rM_{z+1} + tN_{z+1} + 1. \tag{11}$$

Similarly

$$M_z = rN_{z-1} + tM_{z-1} + 1. \tag{12}$$

We take the boundary conditions to be

$$M_1 = N_{a-1} = 1. \tag{13}$$

Uncoupling (11) and (12) as before we obtain

$$N_z - 2N_{z-1} + N_{z-2} + \frac{2r}{t} = 0. \tag{14}$$

The homogeneous equation corresponding to (14) is identical to (6) and it can readily be verified that $-z^2r/t$ is a particular solution of the complete equation.

Therefore

$$N_z = E + Fz - \frac{z^2 r}{t}. \tag{15}$$

Substituting (15) into (11) we have

$$M_z = E + F\left(z - \frac{1}{r}\right) - \frac{r^2z^2 - 2rz + 1}{rt}.$$

The boundary conditions yield $E=a$ and $F=(ar-1)/t$. Thus

$$N_z = a + \frac{z}{t}(ar - zr - 1), \qquad M_z = a + \frac{(zr - 1)(a - z)}{t}.$$

4. Conclusion. By direct calculation one can verify that the probability that the particle is absorbed at a, given that its initial movement is to the right or left is $1-R_z$, $1-L_z$ respectively. Thus the probability of an endless walk is zero. This is to be expected since for a finite Markov chain with absorbing states the probability is 1 that the chain will end.

For the case $a \to \infty$, that is for a semi infinite walk we find that $L_z = R_z = 1$ and $M_z = N_z = 1$ unless $r=1$ or $z=1$.

Reference

1. W. Feller, An introduction to probability theory and its applications, 2nd ed., Wiley, New York, 1957.

A NOTE ON CONVEX POLYGONS INSCRIBED IN OPEN SETS

ANDREW BRUCKNER, University of California, Santa Barbara

In some recent work concerning a generalization of convex sets, the author has noted a property of bounded open sets which seems to be of independent interest; namely, that a triangle can be inscribed in an arbitrary bounded open set. The purpose of this note is to prove this result and show we cannot, in general, inscribe a convex n-gon in an arbitrary open set if $n > 3$.

Recall that an open set G is called connected provided any two points in G can be joined by a polygonal line lying entirely in G. A standard fact about the euclidean plane E_2 is that any open subset of E_2 can be decomposed, in a unique fashion, into a finite or denumerable number of connected open subsets called components. These components are pairwise disjoint; that is, no two components overlap. In addition, they are maximal, which means that a component of G cannot be properly contained in a connected subset of G. For these and other related results about components, the reader is referred to Apostol [1], page 182.

DEFINITION. *Let G be a bounded open set. The n-gon P is said to be inscribed in G provided its vertices lie on the boundary of G and the rest of P, together with its interior, lies in G.*

Therefore

$$N_z = E + Fz - \frac{z^2 r}{t}. \tag{15}$$

Substituting (15) into (11) we have

$$M_z = E + F\left(z - \frac{1}{r}\right) - \frac{r^2 z^2 - 2rz + 1}{rt}.$$

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DEFINITION. *Let G be a bounded open set. The n-gon P is said to be inscribed in G provided its vertices lie on the boundary of G and the rest of P, together with its interior, lies in G.*

In the proof of Theorem 1 below, we shall make use of the following theorem of Motzkin [**2**], [**3**].

MOTZKIN'S THEOREM. *Let S be a closed subset of the euclidean plane E_2. Then S is convex if and only if every point $p \in E_2$ has a unique nearest point of S.*

For a discussion and proof of this theorem under more restrictive conditions on S, along with another application, the reader may consult [**4**], pages 37–38 and 163–167.

THEOREM 1. *Let G be any bounded, open set in the euclidean plane E_2 and let K be any component of G. Then there exists a triangle inscribed in K.*

Proof. Let p be a point of K having more than one nearest point of the boundary of K. Such a point exists, because if every point in K had a *unique* nearest point in the boundary of K, then by Motzkin's Theorem, the complement of K would be convex, which is absurd. Thus, let q and r be two points on the boundary of K which are nearest p. The interior of the circle C having center p and going through q and r lies entirely in K. If C contains a third point s of the boundary of K, then the points q, r, and s determine the required triangle. Suppose there is no such third point. Let R_q and R_r be open rays originating at q and r respectively, which lie in the same one of the two half-planes determined by the line containing q and r. We may take these rays in such a way that they do not intersect in K and that an initial segment of each of them lies in the interior of C. (For example, we may take R_q and R_r parallel if q and r do not determine a diameter of C, and "almost" parallel if they do.) Let u and v be the intersections of C with R_q and R_r respectively. Let w be the first point on R_r such that the segment determined by u and w contains a boundary point b of K. The interior of the quadrilateral $qrwu$ lies in K and is convex. It follows that the triangle qrb satisfies the requirements of the theorem.

We now show that this result cannot be extended to convex n-gons for $n>3$.

THEOREM 2. *There exists a bounded open set in E_2 in which no convex n-gon can be inscribed if $n>3$.*

Proof. Let G be any open set whose boundary consists of three strictly convex arcs which are pairwise tangent at their endpoints. It is clear that for $n>3$, any n-gon inscribed in G must have self intersections and can therefore not be convex.

This work was done while the author was under the support of NSF grant GP 1592.

References

1. Tom M. Apostol, Mathematical analysis, Addison-Wesley, Reading, Mass., 1957.

2. T. S. Motzkin, Sur quelques propriétés caractéristiques des ensembles convexes, Rend. Reele Acad. Lincei, Cl. Sci. Fis. Mat. Nat., 21 (1935) 562–567.

3. ———, *ibid.*, pp. 773–779.

4. T. M. Yaglom and V. G. Boltyanskii, Convex figures, Holt, Rinehart and Winston, New York, 1961 (Translated by P. J. Kelly and L. F. Walton).

ANSWERS

A340.

$$[a_0 + a_1 + a_2 + \cdots]^x = e^{x \log S} = 1 + \frac{x \log S}{1!} + \frac{x^2 \log^2 S}{2!} + \cdots,$$

where $S = \sum a_r$. Consequently we have

$$a_0 = 1, \qquad a_2 = \frac{a_1^2}{2!}, \cdots, a_r = \frac{a_1^r}{r!}$$

and $S = e^{a_1}$.

A341. $\phi(k)$ denotes the number of integers smaller than k and prime to it. Hence,

$$\sum_1^n \phi(k)$$

is the total number of relatively prime pairs among the first n integers, the total number of pairs being $\binom{n}{2}$. The limit of the given fraction being

$$\sum_1^n \phi(k) \Big/ \binom{n}{2}$$

it will be the probability that any two integers taken at random be relatively prime. This probability is known to have the value $6/\pi^2$. Hence, the limit of the given fraction is $6/\pi^2$.

A342. Since $\sqrt[n]{n!}$ is the geometric mean of 1, 2, 3, $\cdots$, n, it is obviously an increasing function of n, so

$$\sqrt[9]{9!} < \sqrt[10]{10!}$$

A343. The angle θ is evidently 15°.

(Quickies on page 286)

CORRECTION

In the paper "A Theorem on Integer Quotients of Products of Factorials," this MAGAZINE, **36** (1963) 98, insert between $[l_n] = n$ and Theorem 1 the following sentence:

Assume that if

$$[l_n] = [l_a] + [l_b] + [l_c] + \cdots + [l_k],$$

then

$$\left[\frac{l_n l_m}{l_r}\right] \geqq \left[\frac{l_a l_m}{l_r}\right] + \left[\frac{l_b l_m}{l_r}\right] + \cdots + \left[\frac{l_k l_m}{l_r}\right].$$

A COMPLETE SET OF COEFFICIENT FUNCTIONS FOR THE SECOND DEGREE EQUATION IN TWO VARIABLES

J. M. STARK, Stanford University

1. Introduction. The purpose of this paper is to give a complete theory of coefficient functions for the second degree equation in two variables. By this we mean that we shall define a set of functions on the real coefficients a, h, b, g, f, c of the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \tag{1.1}$$

where a, h, b are not all zero, in such a way that canonical transformation is accomplished conveniently by merely referring values of these functions to a set of tables. (See section 5 below.) Such tables are provided here which give not only the canonical form in every case of a real locus, degenerate or otherwise, but also complete specifications for the transformed axes. The method employed permits the canonical form to be found separately from (even before) the coordinates of the new origin and the disposition of the transformed axes. This material obviously has application in writing automatic computing machine programs to determine the locus represented by an arbitrary equation of the form (1.1).

In the literature (see bibliography provided at end of paper) one encounters the set of coefficient functions

$$L = a + b, \qquad D = ab - h^2, \qquad J = ac + bc - f^2 - g^2,$$

$$M = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

These functions are used in a classification of the curve(s) or point which may be represented by an equation of the form (1.1), but this set of functions does not determine uniquely the locus represented by the equation. This is quickly apparent by an example, if not by other means. Consider, for example, the equations

$$y^2 - 2x = 0, \qquad x^2 - 2y = 0.$$

For each of these we have $L=1$, $D=0$, $J=M=-1$. Any equation of the form (1.1) with these values for L, D, J, M represents a parabola, but, as this example shows, even if all parabolas of such equations are congruent, they do not all coincide when plotted in the (x, y) plane. In this sense, then, the functions L, D, J, M do not characterize the locus represented by an equation (1.1).

The following question arises: Can we join to the set of functions L, D, J, M additional coefficient functions such that tables of convenient size give from values of these functions, and in a simple form, complete specifications for the locus in every case, degenerate or otherwise? We find the answer in the affirmative and provide such tables together with proofs that these tables are complete.

In section 2 below, the coefficient functions used in this paper are defined. The tables given in section 5 provide complete specifications for the locus of a given equation (1.1). The theorem of section 4 shows that the tables provided in this paper are complete.

In applying the procedures described here, it is convenient that only a few of the coefficient functions of section 2 need to be evaluated in any particular case.

2. Definitions of useful coefficient functions. L, J, D, M are as defined above. $H=a^2-b^2$, $S=a(f-g)$, $T=a(f+g)$, $U=(a-h)^2$, $V=(a+h)^2$, $C=(a-b)M$, $P=aM$, $Q=hM$, $N=ah$, $F=\sqrt{(4h^2+(a-b)^2)}$.

For $a-b\neq 0$, we define $R=2h/(a-b)$. We use the symbol $R=\infty$ if and only if $a=b$ and $h\neq 0$.

For R defined and finite we define $p=(1/\sqrt{2})\sqrt{(1+(1+R^2)^{-1/2})}$, $q=(1/\sqrt{2})\sqrt{(1-(1+R^2)^{-1/2})}$.

For $L\neq 0, aL\geqq 0, bL\geqq 0$, we define $A=L(f\sqrt{(b/L)}-g\sqrt{(a/L)}), B=L(f\sqrt{(b/L)}+g\sqrt{(a/L)})$, $Z=L(f\sqrt{(a/L)}-g\sqrt{(b/L)})$, $W=L(f\sqrt{(a/L)}+g\sqrt{(b/L)})$.

All radicals indicate nonnegative square roots. Each of the conditions $R>0$, $R\geqq 0$, $R<0$, $R\leqq 0$ implies that neither $R=\infty$ nor $F=0$ is the case. In correct applications of the procedures described here, we never encounter the square root of a negative number, and division is indicated only under circumstances (perhaps requiring proof) which imply that the divisor is different from zero.

3. Invariant aspects of the coefficient functions. In reducing equation (1.1) to its canonical form, we employ the following notation and types of transformations:

(3.1) Parallel translation of axes with new origin at $x=u$, $y=v$.

(3.2) Rotation of axes through angle θ.

(3.3) Multiplication of equation by nonzero constant k.

Let

$$a'x'^2+2h'x'y'+b'y'^2+2g'x'+2f'y'+c'=0 \tag{3.4}$$

be the result of transforming a given equation (1.1) by a transformation of type (3.1) $x=x'+u$, $y=y'+v$. If Γ is a class of equations of the form (1.1) and $\Omega=\Omega(a, h, b, g, f, c)$ is a coefficient function possessing the property

$$\Omega(a, h, b, g, f, c)=\Omega(a', h', b', g', f', c')$$

for each equation of the class Γ and every transformation of the type (3.1), then Ω is said to be invariant with respect to translation for the class Γ. Similar definitions referring to transformations of types (3.2), (3.3), respectively, define the terms "invariant with respect to rotation for the class Γ", and "invariant with respect to multiplication for the class Γ."

Convenient proofs for information in the tables of this paper are obtained by utilizing invariant aspects of the coefficient functions defined in section 2.

Each of the coefficient functions L, D, J, M is invariant (see bibliography) with respect to rotation for any class of equations (1.1). Each of L, D, M is invariant with respect to translation for any class of equation (1.1).

If Γ is a class of equations of form (1.1) for which R is defined, then R, p, q are invariant with respect to multiplication for class Γ.

If Γ is a class of equations of the form (1.1) for which W is defined and not zero, then the algebraic sign of W is invariant with respect to multiplication for the class Γ. The last statement holds also when W is replaced by each of J, D, H, S, T, U, V, C, P, Q, N, F, A, B, Z. Thus each of the coefficient functions defined in section 2 possesses some invariant aspect with respect to one or more of the operations (3.1), (3.2), (3.3) used in canonical reduction.

Classification for determining the locus of an equation of the form (1.1) is based upon algebraic signs of coefficient functions whose algebraic signs exist and are invariant with respect to multiplication for a class of equations, including the equation under consideration. Canonical forms are calculated in terms of coefficient functions invariant with respect to all of translation, rotation, and multiplication for a class Γ of equations (1.1) such that Γ contains the equation under consideration.

Coefficient functions of this latter type are called absolute invariants for the class Γ. For example, $[-(LD/M)+|D/M|F]$ is an absolute invariant with respect to that class of equations (1.1) distinguished by the property $M\neq 0$. (See section 5 below.) Other absolute invariants for various classes of equations (1.1) are found by examining the canonical forms given below.

If $\Omega=\Omega(a, h, b, g, f, c)$ is a coefficient function defined for a class Γ of equations (1.1), and if $n=\text{integer}\geqq 0$, then Ω is said to be an invariant of order n for the class Γ if and only if:

(3.5) Ω is invariant with respect to both translation and rotation for the class Γ, and

(3.6) Ω is a homogeneous function of order n. (I.e., $\Omega(ka, kh, kb, kg, kf, kc) = k^n\Omega(a, h, b, g, f, c)$ for every constant k.)

For example, L, D, M are invariants of orders 1, 2, 3, respectively, for every class of equations of the form (1.1).

4. Situations which do not occur. Because of the functional relationships between the coefficient functions, defined in section 2, certain conditions on certain sets of them cannot occur. In this paper we consider only the case in which a, h, b, are not all zero. When we say, for example, that $D=L=0$ does not occur, we mean that $D=L=0$ is inconsistent with the relation $a^2+h^2+b^2\neq 0$.

The following theorem lists circumstances which we need not consider in any discussion of an equation of the form (1.1). Also, this theorem is used to show that this paper gives complete specifications for the locus (if any) represented by every equation of the form (1.1).

THEOREM. *For equations* (1.1) *and the quantities defined in section 2 above, the situations* (4.1)–(4.15) *listed below do not occur.*

(4.1) $D=0$, $M\neq 0$, $R\geqq 0$, $H<0$, $W=0$

(4.2) $D=0$, $M\neq 0$, $R\geqq 0$, $H>0$, $Z=0$

(4.3) $D=0$, *R defined and finite*, $H=0$

(4.4) $D=0$, $M\neq 0$, $R<0$, $H<0$, $Z=0$

(4.5) $D=0,\ M\neq 0,\ R<0,\ H>0,\ W=0$
(4.6) $D=0,\ M\neq 0,\ R=\infty,\ U=S=0$
(4.7) $D=0,\ R=\infty,\ U\neq 0,\ V\neq 0$
(4.8) $D=0,\ M\neq 0,\ R=\infty,\ V=T=0$
(4.9) $D=0,\ F=0$
(4.10) $D>0$, *R defined and finite*, $H=0$
(4.11) $D>0,\ R=\infty,\ N=0$
(4.12) $D<0,\ F=0$
(4.13) $M\neq 0,\ R=\infty,\ Q=0$
(4.14) $M\neq 0$, *R defined and finite*, $C=0$
(4.15) $D\geqq L=0$.

To illustrate the methods used to prove this theorem, consider specifically (4.5). Using $D=ab-h^2=0$ in the expression for M, we obtain

$$2fgh \neq bg^2 + af^2. \tag{4.16}$$

$H=a^2-b^2>0$ implies that $L=a+b\neq 0$. $R<0$ gives $h\neq 0$, which, together with $D=0$, shows that $ab\neq 0$. Now $aL=a^2+h^2$, $bL=h^2+b^2$ shows that a, b, L have the same sign, and so W is defined. Since $L\neq 0$, it follows from $W=0$ that $(g\sqrt{(b/L)}+f\sqrt{(a/L)})^2=0$, or

$$-2Lfg\sqrt{(ab/L^2)} = bg^2 + af^2. \tag{4.17}$$

Now $R<0$ implies that h, $(a-b)$ are of opposite sign. $H>0$ gives that L, $(a-b)$ are of the same sign. Hence $\sqrt{(ab/L^2)}=\sqrt{(h^2/L^2)}=-h/L$, which shows that (4.17) is inconsistent with (4.16).

The remaining parts of the theorem follow in a similar manner.

5. Complete specifications in terms of coefficient functions for locus of an equation (1.1). Next we describe a tabular procedure to obtain complete specifications for a real locus represented by an equation of the form (1.1). After this description, methods of proof are considered which utilize the properties of the coefficient functions.

First compute L, D, M. If $D>0$ and $LM>0$, then there is no real locus. If $D=M=0$, compute J. If $D=M=0$ and $J>0$, then there is no real locus. In all other cases which can occur, the locus is real.

Case I: $D>0$, $LM<0$. In this case the locus is the ellipse, which, referred to (X, Y) axes yet to be determined, may be represented

$$(1/\alpha^2)X^2 + (1/\beta^2)Y^2 = 1,$$

where

$$(1/\alpha^2) = (-1/2)[(LD/M) + |D/M|F],\ (1/\beta^2) = (-1/2)[(LD/M) - |D/M|F].$$

The origin of the (X, Y) axes is at $x=(hf-bg)/D$, $y=(hg-af)/D$. In each of the tables which follow, angle θ is measured in the positive sense from the positive x-axis as initial side to the positive X-axis as terminal side.

$D>0$, $LM<0$	$\cos\theta$	$\sin\theta$
$R\geqq 0$, $H<0$	p	q
$R\geqq 0$, $H>0$	$-q$	p
$R<0$, $H<0$	$-p$	q
$R<0$, $H>0$	q	p
$R=\infty$, $N<0$	$1/\sqrt{2}$	$1/\sqrt{2}$
$R=\infty$, $N>0$	$-1/\sqrt{2}$	$1/\sqrt{2}$
$F=0$	1	0

TABLE 1. Ellipse

Case II: $D<0$, $M\neq 0$. In this case, the locus is the hyperbola, which, referred to transformed axes (X, Y), is represented $(1/\alpha^2)X^2-(1/\beta^2)Y^2=1$, where

$$(1/\alpha^2) = (1/2)[-(LD/M) + |D/M| F], \quad (1/\beta^2) = (1/2)[(LD/M) + |D/M| F].$$

The origin of the (X, Y) axes is at $x=(hf-bg)/D$, $y=(hg-af)/D$. The angle θ made by the positive X axis with the positive x axis is given by Table 2.

$D<0$, $M\neq 0$	$\cos\theta$	$\sin\theta$
$R\geqq 0$, $C>0$	p	q
$R\geqq 0$, $C<0$	$-q$	p
$R<0$, $C>0$	$-p$	q
$R<0$, $C<0$	q	p
$R=\infty$, $Q>0$	$1/\sqrt{2}$	$1/\sqrt{2}$
$R=\infty$, $Q<0$	$-1/\sqrt{2}$	$1/\sqrt{2}$

TABLE 2. Hyperbola

Case III: $D=0$, $M\neq 0$. In this case, the locus is the parabola, which, referred to transformed axes (X, Y), is written

$$Y^2 - 2(\sqrt{(-M/L^3)})X = 0.$$

The origin of the (X, Y) axes is at $x=\bar{u}\cos\theta-\bar{v}\sin\theta$, $y=\bar{u}\sin\theta+\bar{v}\cos\theta$, where $\bar{u}$, $\bar{v}$, $\cos\theta$, $\sin\theta$ are given by Table 3. The positive X axis makes the angle θ with the positive x axis.

$D=0,\ M\neq 0$	$\cos\theta$	$\sin\theta$	$\bar{u}$	$\bar{v}$
$R\geqq 0,\ H<0,\ W<0$	p	q	$(A^2-cL^3)/(2L^2W)$	$-A/L^2$
$R\geqq 0,\ H<0,\ W>0$	$-p$	$-q$	$(cL^3-A^2)/(2L^2W)$	A/L^2
$R\geqq 0,\ H>0,\ Z<0$	$-q$	p	$(B^2-cL^3)/(2L^2Z)$	B/L^2
$R\geqq 0,\ H>0,\ Z>0$	q	$-p$	$(cL^3-B^2)/(2L^2Z)$	$-B/L^2$
$R<0,\ H<0,\ Z<0$	$-p$	q	$(B^2-cL^3)/(2L^2Z)$	B/L^2
$R<0,\ H<0,\ Z>0$	p	$-q$	$(cL^3-B^2)/(2L^2Z)$	$-B/L^2$
$R<0,\ H>0,\ W>0$	$-q$	$-p$	$(cL^3-A^2)/(2L^2W)$	A/L^2
$R<0,\ H>0,\ W<0$	q	p	$(A^2-cL^3)/(2L^2W)$	$-A/L^2$
$R=\infty,\ U=0,\ S>0$	$1/\sqrt{2}$	$-1/\sqrt{2}$	$(4a^3c-T^2)/(4\sqrt{2}a^2S)$	$-T/(2\sqrt{2}a^2)$
$R=\infty,\ U=0,\ S<0$	$-1/\sqrt{2}$	$1/\sqrt{2}$	$(T^2-4a^3c)/(4\sqrt{2}a^2S)$	$T/(2\sqrt{2}a^2)$
$R=\infty,\ V=0,\ T<0$	$1/\sqrt{2}$	$1/\sqrt{2}$	$(S^2-4a^3c)/(4\sqrt{2}a^2T)$	$-S/(2\sqrt{2}a^2)$
$R=\infty,\ V=0,\ T>0$	$-1/\sqrt{2}$	$-1/\sqrt{2}$	$(4a^3c-S^2)/(4\sqrt{2}a^2T)$	$S/(2\sqrt{2}a^2)$

TABLE 3. Parabola

Case IV: $D<0,\ M=0$. In this case the locus consists of intersecting straight lines, which, referred to transformed axes $(X,\ Y)$ is represented

$$(Y-mX)(Y+mX)=0, \quad \text{where} \quad m^2=(-|L|+F)/(|L|+F).$$

The origin of the $(X,\ Y)$ axes is at $x=(hf-bg)/D,\ y=(hg-af)/D$. The angle θ made by the positive X axis with the positive x axis is given by Table 4.

$D<0,\ M=0$	$\cos\theta$	$\sin\theta$
$R\geqq 0,\ H>0$	$-q$	p
$R\geqq 0,\ H\leqq 0$	p	q
$R<0,\ H>0$	q	p
$R<0,\ H\leqq 0$	$-p$	q
$R=\infty,\ N\leqq 0$	$1/\sqrt{2}$	$1/\sqrt{2}$
$R=\infty,\ N>0$	$-1/\sqrt{2}$	$1/\sqrt{2}$

TABLE 4. Intersecting lines

Case V: $D=M=0$, $J\leqq 0$. In this case, the locus consists of parallel or coincident lines, which, referred to transformed axes (X, Y) may be represented

$$(Y - \sqrt{(-J/L^2)})(Y + \sqrt{(-J/L^2)}) = 0.$$

The origin of the (X, Y) axes is at $x=-(ag+fh)/L^2$, $y=-(bf+gh)/L^2$. The angle θ made by the positive X axis with the positive x axis is given by Table 5.

$D=M=0$, $J\leqq 0$	$\cos\theta$	$\sin\theta$
$R\geqq 0$, $H<0$	p	q
$R>0$, $H>0$	$-q$	p
$R<0$, $H<0$	$-p$	q
$R\leqq 0$, $H>0$	q	p
$R=\infty$, $U\neq 0$	$1/\sqrt{2}$	$2/\sqrt{1}$
$R=\infty$, $U=0$	$-1/\sqrt{2}$	$1/\sqrt{2}$

TABLE 5. Parallel or coincident lines

Case VI: $D>0$, $M=0$. In this case, the locus is the point located at $x=(hf-bg)/D$, $y=(gh-af)/D$. Referred to transformed axes (X, Y), the equation reduces to

$$(|L| + F)^2X^2 + 4DY^2 = 0, \quad \text{where} \quad |L| + F \neq 0.$$

The origin of the (X, Y) axes is at this point, and the angle θ made by the positive X axis with the positive x axis is given by Table 6.

$D>0$, $M=0$	$\cos\theta$	$\sin\theta$
$R\geqq 0$, $H>0$	p	q
$R>0$, $H<0$	$-q$	p
$R<0$, $H>0$	$-p$	q
$R\leqq 0$, $H<0$	q	p
$R=\infty$, $N>0$	$1/\sqrt{2}$	$1/\sqrt{2}$
$R=\infty$, $N<0$	$-1/\sqrt{2}$	$1/\sqrt{2}$
$F=0$	1	0

TABLE 6. Point

6. Proofs for tabulated data. To demonstrate how the properties of the coefficient functions are used to advantage, consider specifically the case $D<0$ $R>0$, $C<0$. Here $C=(a-b)M<0$ implies $M\neq 0$, and the locus is a hyperbola, for which we use the canonical form having foci on the X axis

$$(1/\alpha^2)X^2 - (1/\beta^2)Y^2 = 1. \tag{6.1}$$

As one of the operations involved in transforming the original equation (1.1) into (6.1), we have multiplied by a nonzero constant k. As a matter of notation, let L, D, M be the coefficient functions of section 2 evaluated on the coefficients of the original equation, and let $\tilde{L}$, $\tilde{D}$, $\tilde{M}$ denote the corresponding functions evaluated on the coefficients of the (transformed) equation (6.1). Then, since L, D, M are invariants of orders 1, 2, 3, respectively, we have

$$\tilde{L} = kL, \qquad \tilde{D} = k^2 D, \qquad \tilde{M} = k^3 M, \quad \text{or}$$

$$(1/\alpha^2) - (1/\beta^2) = kL, \quad -1/(\alpha^2\beta^2) = k^2 D, \quad 1/(\alpha^2\beta^2) = k^3 M. \tag{6.2}$$

Solving (6.2) for $(1/\alpha^2)>0$, $(1/\beta^2)>0$, and making use of the identity $F^2=L^2-4D$, we obtain the expressions of section 5.

Next, we show that the translation and rotation indicated in section 5 and Table 2, actually transforms our original equation (1.1), except possibly for multiplication by a suitable constant, into an equation of the form (6.1). Under the translation $x=x'+u$, $y=y'+v$ equation (1.1) becomes

$$ax'^2 + 2hx'y' + by'^2 + 2x'(au + hv + g) + 2y'(hu + bv + f) + c' = 0, \tag{6.3}$$

where c' is independent of x', y'. Solving for u, v so as to eliminate the x', y' terms, we find u, v to be as in section 5. After this translation we transform the (x', y') axes into (X, Y) axes by rotating through the angle θ given by Table 2. This transforms (6.3) into

$$\hat{a}X^2 + 2\hat{h}XY + \hat{b}Y^2 + c' = 0, \tag{6.4}$$

where

$$\begin{aligned} \hat{a} &= aq^2 - 2hpq + bp^2 = [(a + b) - (a - b)\sqrt{(1 + R^2)}]/2, \text{ and} \\ \hat{h} &= apq + h(q^2 - p^2) - bpq = 0. \end{aligned} \tag{6.5}$$

Here we have made use of the fact that in the case under consideration $R=[2h/(a-b)]>0$.

By the invariant properties of L, M it follows that

$$\hat{a} + \hat{b} = a + b = L, \qquad \hat{a}\hat{b}c' = M,$$

which, together with (6.5) gives

$$2\hat{b} = (a + b) + (a - b)\sqrt{(1 + R^2)}. \tag{6.6}$$

If $L=0$, then $\hat{b}=-\hat{a}=(a-b)\sqrt{(1+R^2)}\neq 0$ since $C=(a-b)M<0$ here. Now

$$\hat{a}^3c' = -\hat{a}M = C\sqrt{(1 + R^2)} < 0$$

shows that a, c' are of opposite sign, and it follows that, except possibly for a constant multiplier, (6.4) is of the form (6.1).

If $L\neq 0$, then $D<0$, $C<0$ implies that $HLM=L^2C<0$. Thus H and LM are of opposite signs. If $LM<0$, then

$$\hat{b}M = [L + (a - b)\sqrt{(1 + R^2)}]M = LM + C\sqrt{(1 + R^2)} < 0$$

implies $\hat{a}c' = M/\hat{b} < 0$, and (6.4), except possibly for a constant multiplier, is of the form (6.1). If $LM>0$, then $H<0$ in this case, and the hyperbola is distinguished by the fact that its transverse axis is shorter than its conjugate axis (refer to formulas of section 5). Now $H<0$ implies that $a+b$, $a-b$ are opposite in sign, which gives $|\hat{a}|>|\hat{b}|$. Hence (6.4) must reduce to the form (6.1) upon multiplication by a suitable constant.

In proof for the parabola case, it appears more convenient to verify the angle of rotation before establishing the vertex. Consider, for example, the case $D=0$, $R<0$, $H<0$, $Z>0$. If follows from these conditions that $M\neq 0$, and the locus is a parabola. After rotation through the angle θ given by Table 3, $\cos\theta=p$, $\sin\theta=-q$, the given equation (1.1) becomes

$$\hat{a}x'^2 + 2\hat{h}x'y' + \hat{b}y'^2 + 2\hat{g}x' + 2\hat{f}y' + c = 0, \tag{6.7}$$

where

$$\hat{a} = ap^2 - 2hpq + bq^2, \qquad \hat{h} = apq + h(p^2 - q^2) - bpq. \tag{6.8}$$

Now $H<0$ implies that $a+b$ and $a-b$ have opposite signs. $R<0$ gives $h(a-b)<0$, and so h and L are the same sign. $D=0$ gives $ab=h^2$, whence $\sqrt{(1+R^2)}=(a+b)/(b-a)$, $pq=h/L$. Using these in (6.8) we obtain $\hat{a}=\hat{h}=0$, and (6.7) appears as

$$\hat{b}y'^2 + 2\hat{g}x' + 2\hat{f}y' + c = 0, \tag{6.9}$$

where

$$\hat{b} = L, \qquad \hat{g} = -Z/L, \qquad \hat{f} = B/L.$$

In this case, $\hat{b}\hat{g}=-Z<0$. Finally, we reduce equation (6.9) to the canonical form of section 5 by applying the translation $X=x'+\tilde{u}$, $Y=y'+\tilde{v}$. This process determines $\tilde{u}$, $\tilde{v}$ to be as tabulated in Table 3. To obtain the vertex coordinates in terms of the original (x, y) coordinates, we use the relations given in section 5.

REMARK. Perhaps a convenient complete system of coefficient functions and accompanying theory can be found for the second degree equation in three variables. If so, such a theory necessarily contains the material discussed here as a special case, just as the tables given here will find the real roots of quadratic equations in one variable.

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COIN TOSSING, PROBABILITY, AND THE WEIERSTRASS APPROXIMATION THEOREM

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1. Introduction. The purpose of this article is to give a full account of the details and the motivation that are behind Bernstein's proof [1] of Weierstrass' famous approximation theorem. Although these matters are not beyond the reach of the undergraduate student in an advanced calculus course, the author knows of no easily accessible treatment that fully exposes the idea of the proof.

2. Statement of the theorem. The theorem of Weierstrass states: *Every function $x(t)$, continuous on an interval $a \leqq t \leqq b$, can be arbitrarily closely and uniformly approximated by polynomials.*

More precisely, for any $\epsilon > 0$ there is a polynomial $P(t)$ such that $|x(t) - P(t)| \leqq \epsilon$ for all t satisfying $a \leqq t \leqq b$.

Since the correspondence $x(t) \leftrightarrow x[a + t(b-a)]$ demonstrates the essential equivalence of functions on $a \leqq t \leqq b$ and functions on $0 \leqq t \leqq 1$ (geometrically, the graph of one is obtained from the graph of the other by the reversible operations of translation and stretching), we do not lose generality if we give a proof of the Weierstrass theorem for the interval $0 \leqq t \leqq 1$ only.

3. Motivation for the proof. Imagine a loaded coin which turns up "heads" with probability t, $0 \leqq t \leqq 1$. In a game consisting of n consecutive tosses, the probability that exactly k "heads" occur is given by

$$\frac{n!}{k!(n-k)!} t^k (1-t)^{n-k}, \qquad (k = 0, 1, 2, \cdots n).$$

Now suppose that the continuous function $x(t)$ assigns a value to the game as follows: the game shall pay off $x(\frac{k}{n})$ dollars if exactly k of the n tosses are "heads." We obtain for the "expected value" E_n (mean value) of one play consisting of n tosses

$$E_n = \sum_{k=0}^{n} V_k P_k,$$

where V_k = pay-off for k heads among n tosses, and P_k = probability of outcome k heads among n tosses. Since

$$V_k = x\left(\frac{k}{n}\right) \quad \text{and} \quad P_k = \frac{n!}{k!(n-k)!} t^k (1-t)^{n-k},$$

we obtain

$$E_n(t) = \sum_{k=0}^{n} x\left(\frac{k}{n}\right)\binom{n}{k} t^k (1-t)^{n-k},$$

for the average value of one play of the game.

Now, if n is very large, it is reasonable to expect that head will come up nearly nt times, since t is the probability for head. This means that $x(tn/n)=x(t)$ and $E_n(t)$ will be nearly the same for large values of n, i.e., we expect the difference $|x(t)-E_n(t)|$ to tend to zero as n increases to infinity. But $E_n(t)$ is a polynomial of nth order in t, so what we have established by a heuristic argument is already Weierstrass' approximation theorem, which we will now proceed to prove by essentially standard methods.

4. A start on the proof. We begin by attempting to estimate the size of the quantity $|x(t)-E_n(t)|$. Using the identity

$$1 = 1^n = [t + (1 - t)]^n = \sum_{k=0}^{n} \binom{n}{k} t^k(1 - t)^{n-k}$$

we can write

$$x(t) = \sum_{k=0}^{n} x(t) \binom{n}{k} t^k(1 - t)^{n-k}$$

and

$$\begin{aligned} |x(t) - E_n(t)| &= \left| \sum_{k=0}^{n} \left[x(t) - x\left(\frac{k}{n}\right) \right] \binom{n}{k} t^k(1 - t)^{n-k} \right| \\ &\leqq \sum_{k=0}^{n} \left| x(t) - x\left(\frac{k}{n}\right) \right| \binom{n}{k} t^k(1 - t)^{n-k}. \end{aligned}$$

We will now utilize the continuity of $x(t)$, and the fact that a function which is continuous on a closed interval is uniformly continuous there, i.e., for any $\epsilon>0$ there is a $\delta(\epsilon)$ such that $|x(t)-x(t')| \leqq \epsilon$ whenever $|t-t'| \leqq \delta(\epsilon)$. In view of this it is natural to divide the values of k into two groups,

$$S_1 = \text{all values of } k \text{ satisfying} \left| t - \frac{k}{n} \right| \leqq \delta(\epsilon)$$

and

$$S_2 = \text{all values of } k \text{ satisfying} \left| t - \frac{k}{n} \right| > \delta(\epsilon).$$

Clearly, $S_1 \cap S_2 = 0$. Then

$$\begin{aligned} |x(t) - E_n(t)| \leqq &\sum_{k\in S_1} \left| x(t) - x\left(\frac{k}{n}\right) \right| \binom{n}{k} t^k(1 - t)^{n-k} \\ &+ \sum_{k\in S_2} \left| x(t) - x\left(\frac{k}{n}\right) \right| \binom{n}{k} t^k(1 - t)^{n-k}. \end{aligned}$$

But $|x(t)-x(k/n)| \leqq \epsilon$ for k in S_1, and hence

$$|x(t) - E_n(t)| \leqq \epsilon \sum_{k\in S_1} \binom{n}{k} t^k(1 - t)^{n-k} + \sum_{k\in S_2} \left| x(t) - x\left(\frac{k}{n}\right) \right| \binom{n}{k} t^k(1 - t)^{n-k}.$$

Now, since a continuous function on a closed interval is bounded, we have $|x(t)| \leqq M$ for $0 \leqq t \leqq 1$ and $|x(t) - x(k/n)| \leqq 2M$. Moreover,

$$\sum_{k \in S_1} \binom{n}{k} t^k (1-t)^{n-k} \leqq \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} = 1,$$

and we obtain

$$(1) \qquad |x(t) - E_n(t)| \leqq \epsilon + 2M \sum_{k \in S_2} \binom{n}{k} t^k (1-t)^{n-k}.$$

5. The Tchebyshev inequality and the completion of the proof. Recalling that k is in S_2 if and only if $|t - (k/n)| > \delta(\epsilon)$, or $|k - nt| > n\delta(\epsilon)$, we see that

$$\sum_{k \in S_2} \binom{n}{k} t^k (1-t)^{n-k}$$

is exactly the probability that k, the number of "heads" turning up in one play of n tosses, satisfies the inequality $|k - nt| > n\delta(\epsilon)$. We also notice that nt is, on the average, the number of "heads" we would expect in one play.

Now, if a random variable k has average value a, and if b^2 is the average value of $(k-a)^2$, then the Tchebyshev inequality of elementary probability theory states that

$$(2) \qquad Pr[|k - a| > c] \leqq \frac{b^2}{c^2} \cdot$$

Applying this to our case, we obtain

$$(3) \qquad \sum_{k \in S_2} \binom{n}{k} t^k (1-t)^{n-k} \leqq \frac{b^2}{n^2 \delta(\epsilon)^2} \cdot$$

The value of b^2, i.e., the average of $(k - nt)^2$, is by definition

$$b^2 = \sum_{k=0}^{n} (k - nt)^2 \times [\text{Probability of exactly } k \text{ "heads"}]$$

$$= \sum_{k=0}^{n} (k - nt)^2 \binom{n}{k} t^k (1-t)^{n-k}.$$

We will prove in section 6 that this sum has the value $nt(1-t)$.

We obtain from (3) with $b^2 = nt(1-t)$

$$(4) \qquad \sum_{k \in S_2} \binom{n}{k} t^k (1-t)^{n-k} \leqq \frac{t(1-t)}{n\delta(\epsilon)^2} \leqq \frac{1}{4n\delta(\epsilon)^2}$$

since $t(1-t) \leqq \frac{1}{4}$ for $0 \leqq t \leqq 1$. If we substitute this into (1) we obtain

$$(5) \qquad |x(t) - E_n(t)| \leqq \epsilon + \frac{M}{2n\delta(\epsilon)^2}$$

which can obviously be made smaller than any $\epsilon^* > 0$, by choosing $\epsilon = \epsilon^*/2$

and then n so large that $M/2n\delta(\epsilon)^2 < \epsilon^*/2$. This completes the proof of the approximation theorem.

6. The evaluation of b^2 and a direct proof of (3). If we differentiate the identity $1 = \sum_{k=0}^{n} \binom{n}{k} t^k(1-t)^{n-k}$ we obtain after some formal manipulations

$$nt = \sum_{k=0}^{n} k \binom{n}{k} t^k(1 - t)^{n-k}$$

and if we differentiate again we obtain

$$n(n - 1)t^2 = \sum_{k=0}^{n} k(k - 1) \binom{n}{k} t^k(1 - t)^{n-k}.$$

Combination of these three identities yields

$$nt(1 - t) = \sum_{k=0}^{n} (k - nt)^2 \binom{n}{k} t^k(1 - t)^{n-k} = b^2. \tag{6}$$

A discussion of the Tchebyshev inequality which is general enough for our purposes is contained in [2]. However, a direct proof of (3) can be easily obtained from (6) as follows: if we sum in (6) over only those values of k which are in S_2, we obtain

$$\sum_{k\in S_2} (k - nt)^2 \binom{n}{k} t^k(1 - t)^{n-k} \leqq nt(1 - t).$$

Now, for all k in S_2, we have $|k-nt| > n\delta(\epsilon)$ and $(k-nt)^2 > n^2\delta(\epsilon)^2$. Hence the left member of the above inequality is greater than

$$n^2\delta(\epsilon)^2 \sum_{k\in S_2} \binom{n}{k} t^k(1 - t)^{n-k}$$

and we obtain

$$n^2\delta(\epsilon)^2 \sum_{k\in S_2} \binom{n}{k} t^k(1 - t)^{n-k} \leqq nt(1 - t),$$

from which (3) follows.

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NOTE ON MODULES

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It is well known and not too difficult to show (see [3]) that for a left (right) vector space over a division ring, there exists a basis, and all bases have the same cardinality. The same fact about bases is no longer true in a general module over an arbitrary ring. Even if it has bases, in case these bases are finite they do not necessarily have the same cardinality. Everett in [1] and Leavitt in [4] have constructed examples. Consider the following "simpler" example.

Let **V** be a left vector space over a division ring D, with a countable infinite basis $\{a_i\}$. Let R be the ring of endomorphisms of **V** into **V**, where addition is the obvious one and product is defined as composition of mappings. R is a left module over itself, and has a basis of a single element namely the identity endomorphism. But $\{u_1, u_2\}$ also form a basis, where we define for all $n=1, 2, \cdots$

$$a_{2n-1}v_1 = 0 \qquad a_{2n}v_1 = a_n$$
$$a_{2n-1}v_2 = a_n \qquad a_{2n}v_2 = 0$$
$$a_n u_1 = a_{2n} \qquad a_n u_2 = a_{2n-1}$$

where u_i and v_i $(i=1, 2)$ are members of R. Then $e=v_1u_1+v_2u_2$ is clearly the identity endomorphism, and hence $\{u_1, u_2\}$ are generators for R. The R-modules Ru_1 and Ru_2 form a direct sum, since if $z \in Ru_1 \cap Ru_2$ then $z=du_1=bu_2$. But since $u_1v_2=u_2v_1=0$, this implies $z=ze=0$. Thus $\{u_1, u_2\}$ is an independent set of generators, and therefore a basis. Similarly we can also construct a basis of n elements.

Nevertheless if an R-module M, for arbitrary ring R, has an infinite basis, all bases will be infinite and with the same cardinality. This fact is known (follows from results in [2]). The following is an elementary and straightforward proof:

THEOREM. *If an R-module has an infinite basis, then all its bases have the same cardinality.*

Proof. Let M be an R-module with bases $\{u_i\}$, $i \in I$ and $\{v_j\}$, $j \in J$, where I and J are index sets. Assume that the cardinality I is infinite. Write

$$v_j = \sum c_{ij}u_i \qquad \text{for all } j \text{ in } J. \tag{1}$$

This immediately shows that every u_i appears in the right hand side of (1) for if not we can write the one not appearing, say u_k, in terms of the v_j's and hence in terms of the remaining u_i's, contradicting the independence of the u_i's. Now associate with each v_j the subset S_j of the $\{u_i\}$ given by its representation. Each u_i appears in at least one of the S_j, as remarked above. Hence cardinality J is infinite, for otherwise the collection of all sets S_j would be finite, since every S_j is finite. But each u_i is in at least one S_j, contradicting cardinality I infinite.

Let c be the cardinality of the set of subsets S_j. Then $c \leq$ cardinality J, since some of these sets may be repeated. From set theory we have that the cardinality of an infinite union of disjoint, nonempty finite sets is equal to the cardinality of the set of sets. Let S be the disjoint union of the above sets (any repeated u_i is regarded as a separate object); then cardinality $S=c$. But $\{u_i\}\subseteq S$, so cardinality $I\leqq$ cardinality S. Thus cardinality $I\leqq$ cardinality J and, by symmetry, cardinality $J\leqq$ cardinality I. Hence cardinality $I=$ cardinality J.

References

1. C. J. Everett, Jr., Vector spaces over rings, Bull. Amer. Math. Soc., 48 (1942) 312–316.
2. T. Fujiwara, Note on the isomorphism problem for free algebraic systems, Proc. Japan Acad., 31 (1955) 135–6.
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4. W. G. Leavitt, Modules over rings of words, Proc. Amer. Math. Soc., 7 (1956) 188–193.

TWISTED DETERMINANTS THAT SUM TO ZERO

JOEL E. COHEN, Harvard University

Introduction. If $A=|a_{ji}|$ is a square matrix over the reals ($j=1,\cdots,n$, $i=1,\cdots,n$), and $V_j=(a_{j1},\cdots,a_{jn})$ is the jth row vector in A, let $s_j^1(V_j)=(a_{jn},a_{j1},\cdots,a_{j,n-1})$, $s_j^2(V_j)=(a_{j,n-1},a_{jn},a_{j1},\cdots,a_{j,n-2})$, etc. In general, for $1\leqq r\leqq n$, $s_j^r(a_{j1},\cdots,a_{jn})=(X_{j1}^{(r)},\cdots,X_{jn}^{(r)})$. To express $X_{ji}^{(r)}$ as a function of the original components of V_j, we introduce the following function e of any two natural numbers a and b; it is adapted from the ϵ in [2].

$$e(a{:}b)=1 \quad \text{if } a\geqq b, \qquad (1)$$
$$=0 \quad \text{if } a<b.$$

Then

$$X_{ji}^{(r)}=a_{j,i-r+n\cdot e(r:i)}. \qquad (2)$$

Setting

$$I(r)=i-r+n\cdot e(r{:}i), \qquad (3)$$

we have $X_{ji}^{(r)}=a_{j,I(r)}$. By $s_j^r(A)$ we mean the matrix formed from A by applying s_j^r to the jth row vector and leaving all other rows as in A. Similarly, $s_j^r\times s_{j'}^{r'}(A)$ $(j\neq j')$ means the matrix formed by applying s_j^r to V_j, $s_{j'}^{r'}$ to $V_{j'}$, and leaving all other rows the same; and so on. We may thus specify exactly how a matrix is twisted. A twisted determinant is the determinant of the twisted matrix. The purpose of this note is to state a theorem and some corollaries about related twisted determinants that add to zero. These results generalize those stated in [1] and [3].

We recall that if a matrix has just one element a, then the determinant of the matrix is a. The determinant of the empty matrix (having no columns or rows) is 1 (see [2]).

THEOREM. *Let $A=|a_{ji}|$ be an n by n matrix ($n\geqq 2$), and $D(A)$ the determinant of A. Then*

$$\sum_{r=1}^{n} D(s_1^r\times s_2^{n+1-r}(A))=0. \qquad (4)$$

nality of the set of sets. Let S be the disjoint union of the above sets (any repeated u_i is regarded as a separate object); then cardinality $S=c$. But $\{u_i\}\subseteq S$, so cardinality $I\leqq$ cardinality S. Thus cardinality $I\leqq$ cardinality J and, by symmetry, cardinality $J\leqq$ cardinality I. Hence cardinality $I=$ cardinality J.

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Setting

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$$\sum_{r=1}^{n} D(s_1^r\times s_2^{n+1-r}(A)) = 0. \tag{4}$$

Proof. Let $A(1, 2|c_1, c_2)$ be the matrix formed from A by removing rows 1 and 2 and columns c_1 and c_2. Then by Laplace's expansion,

$$D(A) = \sum_{\substack{c_1=1 \\ c_2 \neq c_1}}^{n} \pm \begin{vmatrix} a_{1,c_1} & a_{1,c_2} \\ a_{2,c_1} & a_{2,c_2} \end{vmatrix} D(A(1, 2 | c_1, c_2)).$$

The left-hand side of (4) equals

$$\begin{aligned} &\sum_{r=1}^{n} \sum_{\substack{c_1=1 \\ c_2 \neq c_1}}^{n} \pm \begin{vmatrix} X_{1,c_1}^{(r)} & X_{1,c_2}^{(r)} \\ X_{2,c_1}^{(p)} & X_{2,c_2}^{(p)} \end{vmatrix} D(A(1, 2 | c_1, c_2)) \\ (5) \qquad &= \sum_{\substack{c_1=1 \\ c_2 \neq c_1}}^{n} \pm \left(\sum_{r=1}^{n} \begin{vmatrix} X_{1,c_1}^{(r)} & X_{1,c_2}^{(r)} \\ X_{2,c_1}^{(p)} & X_{2,c_2}^{(p)} \end{vmatrix} \right) D(A(1, 2 | c_1, c_2)), \end{aligned}$$

where

$$(6) \qquad p = n + 1 - r.$$

If we can show that the coefficient in (5) of $D(A(1, 2|c_1, c_2))$ is zero, for every (c_1, c_2), $c_1 \neq c_2$, then we have proved (4). Let c_1, c_2 be fixed at $c_1 = i$, $c_2 = j$, $1 \leqq i, j \leqq n$, $i \neq j$; i and j are fixed for the remainder of the proof. In analogy to (3), let

$$(7) \qquad J(r) = j - r + n \cdot e(r:j).$$

Obviously $I(p) = i - p + n \cdot e(p:i)$ and $J(p) = j - p + n \cdot e(p:j)$. Then by (2)

$$\begin{aligned} (8) \qquad \sum_{r=1}^{n} \begin{vmatrix} X_{1,i}^{(r)} & X_{1,j}^{(r)} \\ X_{2,i}^{(p)} & X_{2,j}^{(p)} \end{vmatrix} &= \sum_{r=1}^{n} \begin{vmatrix} a_{1,I(r)} & a_{1,J(r)} \\ a_{2,I(p)} & a_{2,J(p)} \end{vmatrix} \\ &= \sum_{r=1}^{n} a_{1,I(r)} a_{2,J(p)} - \sum_{r'=1}^{n} a_{1,J(r')} a_{2,I(p')}, \end{aligned}$$

a sum of $2n$ terms, where as before $p' = n + 1 - r'$. From (3) it is clear that as $r = 1, \cdots, i-1$, $I(r) = i-1, \cdots, 1$, and as $r = i, \cdots, n$, $I(r) = n, \cdots, i$. Similarly, as r runs from 1 to n, $J(r)$ ranges over 1 to n. Hence for each r there exists exactly one r' such that $I(r) = J(r')$, i.e., such that

$$(9) \qquad i - r + n \cdot e(r:i) = j - r' + n \cdot e(r':j).$$

If (9) implies that $I(p') = J(p)$, then we may rearrange the terms of (8) into pairs whose members differ only in sign. Then (8) becomes 0 and (4) is proved.

We now show that (9) implies $I(p') = J(p)$. From (9) we have

$$(10) \qquad r' = r + j - i + n(e(r':j) - e(r:i)).$$

There are three possibilities: $e(r':j) - e(r:i)$ can equal 0, +1, or −1.

Case I. If $e(r':j) - e(r:i) = 0$, then $r' = r + j - i$ and (6) gives $p' = p - j + i$. Then $I(p') = i - p' + n \cdot e(p':i) = i - p + j - i + n \cdot e(p - j + i:i) = j - p + n \cdot e(p:j)$

$=J(p)$. Here we use the property of e, that if a, b, c are integers, $e(a:b)=e(a+c:b+c)$.

Case II. If $e(r':j)-e(r:i)=1$, then $r'=r+j-i+n$. Using (6), $p'=p-j+i-n$. By exactly the same substitution as in Case I, $I(p')=j-p+n$. If we can show that $e(p:j)=1$, then $I(p')=j-p+n\cdot e(p:j)=J(p)$. Combining $p'=p-j+i-n$ and $p'=n+1-r'$, we get $p=(n-i)+(n-r')+1+j$, or $p>j$. Therefore $I(p')=J(p)$.

Case III. If $e(r':j)-e(r:i)=-1$, then $r'=r+j-i-n$ and $p'=p-j+i+n$. Then $I(p')=j-p=J(p)$ if $e(p:j)=0$. Combining $p'=p-j+i+n$ and $p'=n+1-r'$, we get $p=j+1-(i+r')$. But since $i\geqq 1$, and $r'\geqq 1$, $p<j$. Therefore $I(p')=J(p)$.

Hence all the terms of (8) cancel and the theorem is proved.

COROLLARY I.

$$\sum_{r=1}^{n} D(\overset{r+k}{s_1} \times \overset{p+k}{s_2}(A)) = 0, \qquad k = 1, \cdots n-1. \tag{11}$$

Adding k to r and to p simply rotates the top two rows together, leaving the rest of the matrix stationary. Since $s_j^{n+a}=s_j^a$ for integral a, (11) provides an alternative $n-1$ ways of expressing (4).

COROLLARY II.

$$\sum_{r=1}^{n} D(\overset{r}{s_u} \times \overset{p}{s_v}(A)) = 0, \qquad 1 \leqq u,\, v \leqq n,\, u \neq v. \tag{12}$$

This is immediate, since interchanging rows just changes the sign of the determinant.

COROLLARY III. *The obvious analogs of the theorem and first two corollaries hold for twists of column vectors instead of row vectors.*

References

1. J. E. Cohen, Further properties of third order determinants, this MAGAZINE, 35 (1962) 304.
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THE WAY OF REDEMPTION

M. D. TEPPER, Urbana, Illinois

Señor Bernardo Ruiz of Cádiz had found in Mr. Arnold an exceptionally shrewd man, a banker by profession, and a gentleman devoted as himself to the mystery of the cards. He decided it would add much to an already pleasant evening to share with him his latest amusement. Consequently, the sentimental señor took only the cards with hearts from the deck and arranged them in the fashion he desired, whereupon he took the first card and put it on the bottom of

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The señor shuffled the deck; Arnold cut it. Ruiz put the first card on the bottom of the deck and turned up the second card, the eight of clubs, etc., until all the cards of the deck were out while Arnold recorded them as they came. Arnold had one hour, more than twice four times the amount of time it took him for the hearts, to win the bet. He was to arrange the cards so that they would come out as he had recorded them—signing his own financial death warrant as the smiling señor said. But now his smile changed to blank expression; once dead-locked opinions altered, for Señor Ruiz' pride was crushed. Mr. Arnold had shown in demonstration what the señor previously only suspected but could not believe—he did have a supernatural method at his command. Arnold had supplied the solution in forty-five minutes.

Losing the $75,000 did not mean as much to the señor as losing face. He offered to double the bet to $150,000 if Arnold could repeat his sorcery in thirty-five minutes. Once again the señor shuffled the deck and Arnold cut it. Ruiz began by putting the first card on the bottom and turning the second card up, etc., as before, while Arnold recorded them. But this time Ruiz cheated: as he progressed through the deck he sporadically put two cards, sometimes even three or four, on the bottom when he should have put only one. At other times, when Arnold was busy recording the previous cards (Arnold often waited till nine or ten cards were out before he recorded the group), Ruiz placed two and later three cards up at once. Alas for Arnold, he failed to find the right combination.

Blank looks and smiles once again changed owners as Arnold put the list of cards in his pocket and left the table for the fresh air of the balcony. What Señor Ruiz had said as a jest became a prediction fulfilled as a scream preluded the death which seemed an accident. Mr. Arnold, whose body was never recovered, perhaps had grown dizzy from the excitement, but such explanations were ruled out when it was later discovered he was destitute. The horrible death of Arnold clouded and encroached upon the señor's thoughts even into sleep, and in a doctor's help he sought relief. Thus the Spaniard unburdened himself of the entire incident; and as he left he could be heard to say, "If only I hadn't cheated. If only I hadn't . . . " With this he vanished from the doctor's sight. (The reader, if he so desires, may now attempt to solve the psychiatrist's dilemma: wherein lies the redemption of Señor Bernardo Ruiz?)

The doctor, as he futilely thought of a means of helping the conscience-stricken man, became desperate; and in doing so he began to investigate each item of his patient's story in the greatest detail. As one might expect, the psychiatrist naturally investigated the most seemingly obscure aspect of the problem first. He wondered how Arnold had solved the problem of hearts, and being something of a mathematician, he soon noticed that the problem was one of permutations—the dealing of the cards from any given arrangement induced a permutation of the cards in the deck into the turned up cards (in the order in which they were turned up). This permutation he called Q. In the case of the hearts, the problem was to find a permutation, or arrangement, P for the cards so that when Q was applied to P, the cards would be in their "natural order," 1, 2, 3, · · · , J, Q, K. Surely! Mr. Arnold had a way of finding P such that $PQ=I$, the identity permutation. Musing further, the psychiatrist realized that in the case of 52 cards, if Mr. Arnold could compute P, then regardless of what permutation R was recorded by him, all he had to do to obtain the arrangement S so that $SQ=R$, was to apply P to R. Then since $PQ=I$, and since $SQ=(RP)Q=R(PQ)$, it would follow that $SQ=R$. Since S depended only on what Mr. Arnold had observed—R, and on P, which he could compute knowing Q, then Ruiz's skullduggery would have had no effect on the outcome.

The psychiatrist thus realized Ruiz' self-reproachment was groundless and that his manipulation of the cards had no effect upon Arnold's losing the bet—Ruiz was not to blame after all for Arnold's suicide. The doctor hurried to Ruiz' villa, but was forced to wait in the study for the gentleman's presence. He did not have long, for what started with hearts came full circle: as the doctor began to tell Ruiz of his discovery, a coronary claimed the Spaniard's life.

GRAPHS OF SEMI-COMPLEX FUNCTIONS

PAUL SCHAEFER, State University of New York at Albany

Recently, Karst [3] described semi-complex functions and their graphs. The following remarks serve to augment this discussion by approaching the subject from a somewhat different point of view.

DEFINITION. *Let D be the domain of the single-valued function f of a complex variable z and let $E=[z:z\in D$ and $f(z)$ is real$]$. When E is nonempty, let f^* be the restriction of f to E. f^* is called the semi-complex function corresponding to f.*

If $f(z)=|z|$, then $f^*=f$, while if $g(z)=i(1+|z|)$, there is no semi-complex function corresponding to g. In general, writing $z=x+iy$ and $f(z)=u(x, y)+iv(x, y)$, one sees that $E=[z:z\in D,\ z=x+iy$ and $v(x, y)=0]$.

Karst notes that the graph of $w=f^*(z)$ may be considered as a subset of the euclidean 3-space, where $P(x, y, w)$ is on the graph if and only if $w=f^*(z)$ and $z=x+iy$. The following result is an immediate consequence of the definition of the graph of f^*.

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THEOREM 1. *Let f^* be the semi-complex function corresponding to* $f(z)=u(x, y)+iv(x, y)$. Let $E^*=[(x, y, w):x+iy\in E, -\infty<w<+\infty]$. *The graph of* $w=f^*(z)$ *is given by the intersection of* E^* *with the surface* $w=u(x, y)$.

When E is a curve or set of curves in the xy-plane, clearly the graph of $w=f^*(z)$ will be the intersection of the cylinder $v(x, y)=0$ and the surface $w=u(x, y)$. If the function f is such that $f(z)$ is real for all real z in its domain, then the graph of $w=f^*(z)$ contains the graph of $w=f(x)$ in the xw-plane. If E coincides with the domain of f, then the graph of $w=f^*(z)$ is the surface $w=u(x, y)$.

It is of interest to note that this last possibility cannot occur with nonconstant analytic functions. For simplicity, the following discussion is restricted to functions which are analytic in the whole complex plane.

THEOREM 2. *If f is a nonconstant entire function and if f^* exists, then E is closed, nowhere dense in the complex plane, and has no isolated points.*

Proof. (1) Let $z_0=x_0+iy_0$ be the limit of a sequence (z_n) of points in E, $z_n=x_n+iy_n$. Now, $v(x_n, y_n)=0$ for every n. Since $v(x, v)$ is continuous, $v(x_0, y_0)=\lim v(x_n, y_n)=0$. This shows that $z_0\in E$ and hence that E is closed.

(2) Since E is closed, in order to show that E is nowhere dense, it suffices to prove that E contains no open sets [1, p. 32]. Suppose that the open set G is a subset of E. Then $v(x, y)=0$ in G. Hence $v_x=v_y=0$ in G. By the Cauchy-Riemann equations, $u_x=u_y=0$ in G and, therefore, f is constant in G. But, according to the Identity Theorem for Analytic Functions [4, p. 87], this means that f is constant everywhere, a contradiction.

(3) Suppose that z_0 is an isolated point of E. Then, $f(z_0)$ is real, and there is a neighborhood N of z_0 which contains no other points of E. Now, analytic functions map neighborhoods onto neighborhoods [2, p. 140], so $f(N)$ is a neighborhood of $f(z_0)$ in the uv-plane which contains no real numbers other than $f(z_0)$. This is impossible, since $f(z_0)$ is a point on the real axis of the uv-plane and thus, every neighborhood of $f(z_0)$ contains other real numbers.

The next theorem shows that, in general, E contains a set of curves.

THEOREM 3. *Let f be a nonconstant entire function; let f^* exist and let* $E'=[z:z\in E \text{ and } f'(z)\neq 0]$. *Then, through every point of E' there passes exactly one curve belonging to E.*

Proof. Let $z_0=x_0+iy_0\in E'$. From the Cauchy-Riemann equations and $f'=u_x+iv_x$ it follows that $|f'(z_0)|^2=v_x^2(x_0, y_0)+v_y^2(x_0, y_0)>0$. Hence, $v_x(x_0, y_0)$ or $v_y(x_0, y_0)$ is different from zero. Suppose that $v_y(x_0, y_0)\neq 0$. Then, by the Implicit Function Theorem [5, p. 56], there exists a unique function $g(x)$ and a positive number c such that $y_0=g(x_0)$ and $v(x, g(x))=0$ for $|x-x_0|<c$. Thus there exists a unique curve, $y=g(x)$, in E which passes through z_0. The case $v_x(x_0, y_0)\neq 0$ is treated similarly.

Points of E which are zeros of $f'(z)$ are singular points of $v(x, y)=0$, and as such, do not lend themselves to a simple discussion. A glance at figures 1, 2, 3, and 5 in [3] shows that such points may be multiple points of the curve $v(x, y)=0$.

The examples given by Karst are entire functions. The above theorems and discussion may help to explain some of the congruent curves obtained in [3] when periodic functions are considered.

Note. The author wishes to acknowledge the suggestions of his colleague, D. Livengood, in the proof of Theorem 2.

References

1. R. P. Boas, A primer of real functions, Carus Math. Monograph No. 13, 1960.
2. C. Caratheodory, Theory of functions of a complex variable, vol. 1. Transl. by F. Steinhardt, Chelsea, New York, 1954.
3. O. Karst, Semi-complex functions and their graphs, this MAGAZINE, 35 (1962) 282–288.
4. K. Knopp, Theory of functions, Part 1. Transl. by F. Bagemihl, Dover, New York, 1945.
5. D. V. Widder, Advanced calculus, 2nd ed., Prentice-Hall, Englewood Cliffs, N. J., 1961.

A PROOF OF THE SUFFICIENCY CONDITION FOR EXACT DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

MORTON J. HELLMAN, Rutgers, The State University

It is well known that $\partial M/\partial y=\partial N/\partial x$ is a necessary and sufficient condition for the differential equation $M(x, y)dx+N(x, y)dy=0$ to be exact. The proof of necessity is trivial and follows immediately from the definition of exactness and the equality of $\partial^2 f/\partial y\partial x$ and $\partial^2 f/\partial x\partial y$ when they are continuous functions of x and y. The proof of sufficiency, however, is not quite so simple and is not handled too well in some of the current books on differential equations. The logical thread of the argument is clouded in a few of the presentations. The following is a proof of the sufficiency condition which approaches the problem in a different fashion.

Let $f(x, y)$, $g(x, y)$ be such that $\partial f/\partial x=M$ and $\partial g/\partial y=N$. Then

$$\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}=\frac{\partial^2 g}{\partial x\partial y}=\frac{\partial^2 g}{\partial y\partial x},$$

the last step following from the assumed continuity of the second partial derivatives. From this,

$$\frac{\partial^2(f-g)}{\partial y\partial x}=0, \quad \text{or} \quad \frac{\partial(f-g)}{\partial x}=f_1(x).$$

Integrating again $f-g=A(x)+B(y)$, where A and B are arbitrary functions of x and y respectively. Hence $f(x, y)=g(x, y)+A(x)+B(y)$. Now

$$\begin{aligned} Mdx+Ndy &= \left[\frac{\partial g}{\partial x}+A'(x)\right]dx+\frac{\partial g}{\partial y}dy \\ &= d[g(x,y)+A(x)] = dF(x,y), \end{aligned}$$

where $F(x, y)=g(x, y)+A(x)=f(x, y)-B(y)$, and the equation $Mdx+Ndy=0$ is exact. By setting $\partial F/\partial x=M$, $\partial F/\partial y=N$, $F(x, y)$ can be determined in the usual way.

The examples given by Karst are entire functions. The above theorems and discussion may help to explain some of the congruent curves obtained in [3] when periodic functions are considered.

Note. The author wishes to acknowledge the suggestions of his colleague, D. Livengood, in the proof of Theorem 2.

References

1. R. P. Boas, A primer of real functions, Carus Math. Monograph No. 13, 1960.
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Let $f(x, y)$, $g(x, y)$ be such that $\partial f/\partial x = M$ and $\partial g/\partial y = N$. Then

$$\frac{\partial^2 f}{\partial y\partial x} = \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \frac{\partial^2 g}{\partial x\partial y} = \frac{\partial^2 g}{\partial y\partial x},$$

the last step following from the assumed continuity of the second partial derivatives. From this,

$$\frac{\partial^2(f-g)}{\partial y\partial x} = 0, \quad \text{or} \quad \frac{\partial(f-g)}{\partial x} = f_1(x).$$

Integrating again $f - g = A(x) + B(y)$, where A and B are arbitrary functions of x and y respectively. Hence $f(x, y) = g(x, y) + A(x) + B(y)$. Now

$$\begin{aligned} M dx + N dy &= \left[\frac{\partial g}{\partial x} + A'(x)\right] dx + \frac{\partial g}{\partial y} dy \\ &= d[g(x, y) + A(x)] = dF(x, y), \end{aligned}$$

where $F(x, y) = g(x, y) + A(x) = f(x, y) - B(y)$, and the equation $Mdx + Ndy = 0$ is exact. By setting $\partial F/\partial x = M$, $\partial F/\partial y = N$, $F(x, y)$ can be determined in the usual way.

PERIODS OF POLYNOMIALS MODULO p

D. E. DAYKIN, Reading University

Many modern machines incorporate electronic circuits which cycle repeatedly through a given sequence of states. Often it is only the length of the cycle which is important and not the nature of the individual states. In certain cases all that is required is that the cycle shall be sufficiently long.

The simplest sequential circuit has a number of elements $U_1, U_2, \cdots, U_n$, each of which has p possible states denoted by $0, 1, \cdots, p-1$, where p is a prime. Further, the state sequence of the elements is determined by the relations

$$u_1' = u_2 - u_1; \qquad u_i' = u_{i+1}\ (1 < i < n); \qquad u_n' = -u_1,$$

where u_i' denotes the state of U_i succeeding state u_i, and arithmetic is performed modulo p. Using polynomial notation we can write the relations as

$$u_1'x + u_2'x^2 + \cdots + u_n'x^n = u_1 + u_2x + \cdots + u_nx^{n-1} - u_1f(x) \pmod p,$$

where $f(x) = 1 + x + x^n$. Hence, if the initial states are given by $u_1 = 1$; $u_i = 0$ $(1 < i \leqq n)$, then the length of the resulting cycle is the least positive integer e such that $f(x)$ divides $x^e - 1 \pmod p$. The integer e is well-known as the period of $f(x)$ and exists for a general polynomial $f(x)$ whenever $f(0) \neq 0$. Moreover if $f(x)$ has degree n and is irreducible modulo p then it has a period e which divides $p^n - 1$ but not $p^r - 1$, $1 \leqq r < n$.

Tables of irreducible polynomials have been produced for the purpose of designing sequential circuits. In general one can say little about the period of a polynomial but we deal here with a case in which the period can be determined. This enables us to design circuits with arbitrarily long cycles.

THEOREM 1. *If m is a positive integer and ϵ is either 0 or 1 then the period of the polynomial*

$$g(x) = x^{p^m+\epsilon} + x + 1,$$

is $p^{2m} - 1$ when $\epsilon = 0$ and $p^{2m} + p^m + 1$ when $\epsilon = 1$.

Proof. We begin by defining an array $(a_{i,j})$, $1 \leqq i, j \leqq p^m$, of integers modulo p. We put

$$\text{(1)} \qquad a_{ij} = (-1)^{i+j} \quad \text{if} \quad i = 1 \quad \text{or} \quad j = 1,$$

and then determine the remaining terms by the relation

$$\text{(2)} \quad a_{ij} = -a_{i+1,j} - a_{i+1,j-1} \pmod p \qquad \text{for } 1 \leqq i \leqq p^m - 1, \quad 2 \leqq j \leqq p^m.$$

Since p divides the binomial coefficients $\binom{p^m}{i}$, $2 \leqq i \leqq p^m - 1$, the array is part of Pascal's triangle, reduced modulo p, with signs changed as shown.

$$\begin{array}{cccccccc}
1 & -1 & 1 & -1 & \cdots & 1 & -1 & 1 \\
-1 & 2 & -3 & 4 & \cdots & -(p^m-2) & (p^m-1) & 0 \\
1 & -3 & 6 & -10 & \cdots & 1 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
-1 & (p^m-1) & 0 & 0 & \cdots & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0
\end{array}$$

The signs may appear to be incorrect for $p=2$, but $-1=+1 \pmod 2$.

Next let $R_i(x)$ be the polynomial whose coefficients form the ith row of the array

$$R_i(x) = a_{i1} + a_{i2}x + \cdots + a_{i,p^m}x^{p^m-1}, \qquad 1 \leqq i \leqq p^m.$$

Then it follows from (1) and (2) that $R_i(x)=-(1+x)R_{i+1}(x)$, $1\leqq i\leqq p^m-1$, and hence that

$$R_i(x) + g(x)R_{i+1}(x) = x^{p^m+\epsilon}R_{i+1}(x), \qquad 1 \leqq i \leqq p^m - 1. \tag{3}$$

Now in order to establish our theorem we need only divide x^o, step by step, by $g(x)$ until the remainder is $-x^e$. The first stage of the division may be written in the form

$$1 = g(x)\{1 - x + x^2 - \cdots - x^{p^m+2\epsilon-2}\} - x^{p^m+3\epsilon-1}R_1(x).$$

It then follows from (3) that the polynomials

$$x^{ip^m+(i+2)\epsilon-1}R_i(x), \qquad 1 \leqq i \leqq p^m,$$

each appear in turn as remainders in the division. The array shows that the last polynomial is in the required form and the theorem is proved.

COROLLARY. *The polynomial $g(x)$ is irreducible only when*

$$p = 2 \quad \textit{and} \quad p^m + \epsilon = 2,\ 3,\ 4, \quad \textit{or} \quad 9.$$

Proof. The least positive integer t such that $p^{2m}+p^m+1$ divides p^t-1 is $3m$. Hence if $\epsilon=1$ and $g(x)$ is irreducible then $p^m+1=3m$. Similarly if $\epsilon=0$ and $g(x)$ is irreducible then $p^m=2m$. The result now follows by inspection of a few simple cases.

The degrees of the irreducible factors of $g(x)$ have been determined in [1].

Reference

1. D. E. Daykin, The irreducible factors of $(cx+d)x^{q^m}-(ax+b)$ over $GF(q)$, Quart. J. Math., Oxford Ser., (2) 14 (1963) 61–64.

PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and exactly the size desired for reproduction.

Send all communications for this department to Robert E. Horton, Los Angeles City College, 855 North Vermont Avenue, Los Angeles 29, California.

PROPOSALS

558. *Proposed by David L. Silverman, Beverly Hills, California.*

Players A and B each has a die, which he places as he sees fit on a table top, without seeing his opponent's play. Simultaneously, the two dice are shown, and the total of the upper faces determines the winner. A wins if the total is a prime; B wins otherwise. Over a long period of time, whom does the game favor?

559. *Proposed by Gilbert Labelle, Université de Montréal.*

Show that

$$\int_0^\infty \frac{dx}{1 + x^\pi} = \frac{1}{\sin 1}\,.$$

560. *Proposed by Morton Hackman, University of Washington.*

Show that the perimeter of a triangle inscribed in a circle is at least twice the diameter of the circle if the triangle contains the center of the circle.

561. *Proposed by Benjamin B. Sharpe, State University of New York at Buffalo.*

Prove that $a^2+b^2=c^2$, $(a\leqq b<c)$ is impossible if a, b, and c are Fibonacci numbers.

562. *Proposed by J. W. Moon and L. W. Beineke, University of London.*

A simple graph G with n points and e edges has the property that of every four points belonging to G some three of these form a triangle. How large must e be for this to be possible?

563. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Let A, B', A', B be four consecutive vertices of a regular hexagon. If M is an arbitrary point of the circumcircle (in particular on arc $A'B'$) and MA, MB intersect BB' and AA' in the points E and F respectively, then prove that:

(a) $\sphericalangle MEF = 3\sphericalangle MAF$

(b) $\sphericalangle MFE = 3\sphericalangle MBE.$

564. *Proposed by Murray R. Spiegel, Rensselaer Polytechnic Institute.*

Show that

$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{(1 - 1/2 \sin^2 \theta)}} = \frac{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12\cdot 14\cdot 16 \cdots}{5\cdot 5\cdot 9\cdot 9\cdot 13\cdot 13\cdot 17\cdot 17 \cdots}.$$

SOLUTIONS

Late Solutions

508. *Walter Meyer, Queens, New York;* **530.** *John W. Milsom, Texas A and I, Kingsville, Texas;* **531.** *Lewis B. Robinson, Baltimore, Maryland.*

Comment on Problem 525

525. [September 1963 and March 1964]. *Proposed by Francis L. Miksa, Aurora, Illinois.*

Comment by Alan Sutcliffe, Knottingley, Yorkshire, England.

With the help of Elizabeth A. Simmonds, programs were written in ALGOL for the Elliott 803 computer to evaluate the coefficient for $n=5$ and $n=6$. For $n=5$, the value of 1,394 was confirmed, and for $n=6$, the value was given as 32,134.

Extreme Overlap

537. [January, 1964]. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Determine the relative positions of an equilateral triangle and a square inscribed in the same circle so that their common area shall be an extremum.

Solution by Michael Goldberg, Washington, D. C.

The extrema are the symmetric relative positions of the square and the triangle.

The minimum overlap occurs when a side of the square is parallel to a side of the triangle as shown in Figure 1. The protruding portions of the triangle are marked A and B. For an infinitesimal rotation from this position, an increase in one of the B areas is compensated by the corresponding decrease in the other B, while the area A is reduced.

The maximum overlap occurs when a vertex of the square coincides with a vertex of the triangle as shown in Figure 2. The equal protruding portions of the triangle are marked C. For an infinitesimal rotation, an increase in one C is compensated by a corresponding decrease in the other C, while a portion of the triangle at the third vertex will now protrude.

If the radius of the circle is unity, the areas are given as follows:

$$A = (2 - \sqrt{2})/2$$

$$2B = \sqrt{3}(\sqrt{3} - \sqrt{2})^2/2$$

$$2C = (9 - 5\sqrt{3})/4.$$

Hence,

$$A + 2B = 0.0933, \text{ maximum protrusion, minimum overlap,}$$
$$2C = 0.0849, \text{ minimum protrusion, maximum overlap.}$$

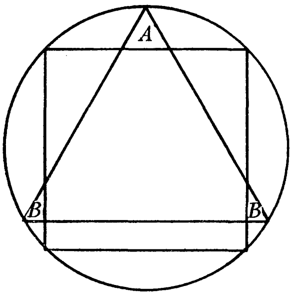

FIG. 1.

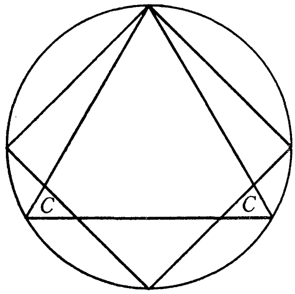

FIG. 2.

An Inequality

538. [January, 1964]. *Proposed by Andrzej Makowski, Warsaw, Poland.*

Prove that for every integer $n>1$, the inequality $[d(n)]^2\phi(n)>\sigma(n)$ holds where $d(n)$ denotes the number of positive divisors of n, $\sigma(n)$ denotes their sum, and $\phi(n)$ is Euler's totient function.

Solution by A. M. Vaidya, Pennsylvania State University.

As all the three functions $d(n)$, $\phi(n)$ and $\sigma(n)$ are multiplicative functions of n, we need to prove the inequality only for prime power values of n.

For $n=p^r$, the inequality takes the form

$$p^{r-1}(r + 1)^2(p - 1)^2 > p^{r+1} - 1,$$

or

$$p^{r-1}[r(r + 2)p^2 - 2(r + 1)^2p + (r + 1)^2] > - 1,$$

or since p and r are integers,

$$p^{r-1}[r(r + 2)p^2 - 2(r + 1)^2p + (r + 1)^2] \geqq 0;$$

this will clearly be satisfied if $p \geqq \max\ [\alpha, \beta]$ where α and β are the roots of

$$r(r + 2)x^2 - 2(r + 1)^2x + (r + 1)^2 = 0,$$

i.e., if $p \geqq (r+1)/r$; and this is always true because $(r+1)/r \leqq 2$. This completes the solution.

Also solved by Josef Andersson, Vaxholm, Sweden; J. L. Brown, Jr., Pennsylvania State University; L. Carlitz, Duke University; Martin J. Cohen, Beverly Hills, California; John E. Jean, Jr., General Dynamics, Fort Worth, Texas; Gilbert Labelle, Université de Montréal, Canada; B. Litvack, University of Michigan; Stanton Philipp, Seal Beach, California; and the proposer.

A Triangle Inequality

539. [January, 1964]. *Proposed by L. Carlitz, Duke University.*

Let P be a point inside the triangle ABC whose distances from the sides are x, y, and z. Let K denote the area and R the circumradius of ABC. Show that $xyz \leqq (2/27)(K^2/R)$ with equality holding only when P is the centroid of ABC.

Solution by J. A. Tyrrell, King's College, London.

We use the well-known fact that the geometric mean of a finite set of positive numbers is not greater than their arithmetic mean, and that the two means are equal if and only if all the numbers of the set are equal. Hence, if α, β, γ are three positive numbers such that $\alpha+\beta+\gamma=1$, their arithmetic mean is $1/3$ and so their geometric mean $\sqrt[3]{\alpha\beta\gamma}$ cannot exceed $1/3$; thus $\alpha\beta\gamma \leqq 1/27$, with equality holding if and only if $\alpha=\beta=\gamma=1/3$.

Now let a, b, c denote the lengths of the sides of ABC; then we have

$$ax + by + cz = 2(\text{area } PBC + \text{area } PCA + \text{area } PAB) = 2K.$$

Thus $ax/2K$, $by/2K$, $cz/2K$ are three positive numbers whose sum is 1. Hence, by the result stated above,

$$\left(\frac{ax}{2K}\right)\left(\frac{by}{2K}\right)\left(\frac{cz}{2K}\right) \leqq \frac{1}{27} \tag{1}$$

with equality if and only if $ax=by=cz$; that is, if and only if

$$\text{area } PBC = \text{area } PCA = \text{area } PAB$$

in which case P is the centroid of ABC.

Now, by elementary trigonometry, $K=\frac{1}{2}ab \sin C$ (area formula) and $2R=c/\sin C$ (sine rule); whence $abc=4KR$. Thus, substituting for abc in (1), we obtain

$$xyz \leqq \frac{2K^2}{27R},$$

with equality if and only if P is the centroid of ABC.

Also solved by Josef Andersson, Vaxholm, Sweden; Leonard D. Goldstone, Watervliet, New York; Stanton Philipp, Seal Beach, California; Hazel S. Wilson, Jacksonville University, Florida; and the proposer.

Multiplication Table

540. [January, 1964]. *Proposed by Leo Moser, University of Alberta.*

Show that if the $n!$ terms in the expansion of an nth order determinant with positive elements a_{ij} have the same absolute value, then there exists a set of numbers $b_1, b_2, \cdots, b_n$ such that $a_{ij}=b_i \cdot b_j$, $i, j=1, 2, \cdots, n$.

Solution by Stanton Philipp, Seal Beach, California.

The statement as it stands is not correct, as is shown by the counter examples

$$\begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix}$$

and

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{vmatrix}$$

However, if we assume that the determinant $|a_{ij}|$ is symmetric, the statement is true.

Let $(p_1, p_2, \cdots, p_{n-2}, r, s)$ and $(i_1, i_2, \cdots, i_{n-2}, r, s)$ be two permutations on the integers $(1, 2, \cdots, n)$. Then $a_{p_1i_1}a_{p_2i_2}\cdots a_{p_{n-2}i_{n-2}}a_{rr}a_{ss} = a_{p_1i_1}a_{p_2i_2}\cdots a_{p_{n-2}i_{n-2}}a_{rs}a_{sr}$, so that $a_{rs} = a_{rr}a_{ss}$. Thus we can take $b_i = \sqrt{(a_{ii})}$, $i = 1, 2, \cdots, n$. Conversely, if $b_i = \sqrt{(a_{ii})}$, then we have $a_{1i_1}a_{2i_1}\cdots a_{n-1r}a_{ns} = (b_1b_2\cdots b_n)^2$.

Also solved by Josef Andersson, Vaxholm, Sweden; L. Carlitz, Duke University; Harry W. Hickey, Computer Dynamics Co., Silver Spring, Maryland; R. H. Hines, Concord, Massachusetts; Gilbert Labelle, Université de Montréal; and the proposer.

The proposer pointed out that another way of stating the condition is to say that the corresponding matrix is a multiplication table.

Ring Probability

542. [January, 1964]. *Proposed by Brother U. Alfred, St. Mary's College, California.*

Given a set of circles of radius R on an extended surface with their centers at the corners of a network of squares of side d. Let a ring of radius r be tossed on the surface. If $r \leqq R$ and $2(R+r) \leqq d$, what is the probability that the ring will touch one of the circles?

Solution by John E. Jean, Jr., General Dynamics, Texas.

We may consider that the surface consists of a network of squares of side d with a circle of radius R centered in each square. Now the center of the ring will land in one of these squares. The portion of the square which, if it contained the center of the ring would permit the ring to touch the circle, is an annulus whose area is given by:

$$A = \pi[(R + r)^2 - (R - r)^2] = 4\pi Rr.$$

Therefore, the desired probability is $4\pi Rr/d^2$.

Also solved by Josef Andersson, Vaxholm, Sweden; Merrill Barneby, University of North Dakota; Harry W. Hickey, Computer Dynamics Co., Silver Spring, Maryland; Michael J. Pascual, Watervliet Arsenal, New York; and the proposer.

Bounds for an Integral

543. [January, 1964]. *Proposed by Murray S. Klamkin, State University of New York at Buffalo.*

If $\int_a^b [F(x)-x^r]^2 dx=\lambda^2$, find upper and lower bounds for $\int_a^b [F(x)]^2 dx$.

(Note: For a class of similar problems, see J. L. Synge, *The Hypercircle in Mathematical Physics*, p. 82.)

Solution by Martin J. Cohen, Beverly Hills, California.

I will prove a more general statement: Let F, G, H be functions such that $F(x)=G(x)+H(x)$. Let

$$A=\left[\int_a^b F^2(x)dx\right]^{1/2},$$

$$B=\left[\int_a^b G^2(x)dx\right]^{1/2},$$

$$C=\left[\int_a^b H^2(x)dx\right]^{1/2},$$

$A\geqq 0$, $B\geqq 0$, $C\geqq 0$. Then $(B-C)^2\leqq A^2\leqq (B+C)^2$.

All we need is the form of the Minkowski integral inequality which states that

$$\left[\int_a^b f^2(x)dx\right]^{1/2}+\left[\int_a^b g^2(x)dx\right]^{1/2}\geqq\left[\int_a^b (f(x)\pm g(x))^2dx\right]^{1/2},$$

$$B+C=\left[\int_a^b G^2(x)dx\right]^{1/2}+\left[\int_a^b H^2(x)dx\right]^{1/2}\geqq\left[\int_a^b (G(x)+H(x))^2dx\right]^{1/2}$$

$$=\left[\int_a^b F^2(x)dx\right]^{1/2}=A,$$

so that $A^2\leqq (B+C)^2$.

$$A+B=\left[\int_a^b F^2(x)dx\right]^{1/2}+\left[\int_a^b G^2(x)dx\right]^{1/2}\geqq\left[\int_a^b (F(x)-G(x))^2dx\right]^{1/2}$$

$$=\left[\int_a^b H^2(x)dx\right]^{1/2}=C$$

and similarly $A+C\geqq B$ so that $A\geqq |B-C|$ and $A^2\geqq (B-C)^2$.

Letting $G(x)=X^r$ we see that

$$B=\left[\frac{b^{2r+1}-a^{2r+1}}{2r+1}\right]^{1/2}$$

so that

$$\int_a^b F^2(x)dx \leqq \left[\left[\int_a^b (F(x) - x^r)^2\right]^{1/2} + \left[\frac{b^{2r+1} - a^{2r+1}}{2r+1}\right]^{1/2}\right]^2$$

and

$$\int_a^b F^2(x)dx \geqq \left[\left[\int_a^b (F(x) - x^r)^2 dx\right]^{1/2} - \left[\frac{b^{2r+1} - a^{2r+1}}{2r+1}\right]^{1/2}\right]^2.$$

Also solved by Michael J. Pascual, Watervliet Arsenal, New York; and the proposer.

Comment on Q263

Q263. [January, 1960]. *Comment by M. S. Klamkin, State University of New York at Buffalo.*

The proof submitted by the proposers, although elegant, is only valid if $p+q$, $q+r$, and $r+p$ form a triangle. The solution $x=\sqrt{(pqr/(p+q+r))}$ is still correct even if a triangle is not formed.

This follows from

$$\arc\tan\frac{p}{x} + \arc\tan\frac{q}{x} + \arc\tan\frac{r}{x} = \arc\tan\frac{x^2(p+q+r) - pqr}{x(x^2 - pq - qr - rp)}.$$

Comment on Q309

Q309. [January 1963]. *Submitted by M. S. Klamkin.*

Comment by Alan Sutcliffe, Knottingley, Yorkshire, England.

There appear to be two compensating errors in this rather abbreviated solution. The first is the assumption that

$$\prod_{n=2}^{\infty} f(n)g(n) = \prod_{n=2}^{\infty} f(n) \prod_{n=2}^{\infty} g(n),$$

which is not true.

The second is in the evaluation of the two products, where cancellation is used to show that

$$\frac{1}{3}\cdot\frac{2}{4}\cdot\frac{3}{5}\cdot\frac{4}{6}\cdot\frac{5}{7}\cdot\cdots = 2,$$

and

$$\frac{7}{3}\cdot\frac{13}{7}\cdot\frac{21}{13}\cdot\frac{31}{21}\cdot\frac{43}{31}\cdot\cdots = \frac{1}{3}.$$

If we replace this second series by

$$\frac{3}{1}\cdot\frac{6}{3}\cdot\frac{10}{6}\cdot\frac{15}{10}\cdot\frac{21}{15}\cdot\cdots$$

($=1$, by the solver's method of cancellation) we can prove that $1=2$, as follows:

$$1 = \prod_{n=2}^{\infty} 1 = \prod_{n=2}^{\infty}\left(\frac{n-1}{n+1}\right)\cdot \prod_{n=2}^{\infty}\left(\frac{\frac{1}{2}n(n+1)}{\frac{1}{2}n(n-1)}\right) = (2)(1) = 2.$$

A valid proof of the original proposition, suggested by the editor, may be given in the following way:

$$\prod_{n=2}^{N}\left[1 - \frac{2}{1+n^3}\right] = \prod_{n=2}^{N}\left(\frac{n-1}{n+1}\right)\cdot \prod_{n=2}^{N}\left(\frac{n^2+n+1}{n^2-n+1}\right)$$

$$= \frac{2}{N(N+1)}\cdot\frac{N^2+N+1}{3}$$

$$= \frac{2}{3}\left(1 + \frac{1}{N^2+N}\right).$$

$$\prod_{n=2}^{\infty}\left[1 - \frac{2}{1+n^3}\right] = \lim_{n\to\infty} \frac{2}{3}\left(1 + \frac{1}{N^2+N}\right) = \frac{2}{3}\,.$$

Comment on Q319

Q319. [September 1963]. *Submitted by C. W. Trigg.*

Comment by M. S. Klamkin, SUNY at Buffalo, New York.

The argument used in obtaining the factorization of Q319, i. e.,

$$a^3 + b^3 + c^3 - 3abc$$

is invalid in general. It works here since the given polynomial, coincidentally, has a pair of symmetric factors. While it is easy to establish that the only factorizations of homogeneous polynomials are into homogeneous polynomials, it is not true that symmetric polynomials factor into symmetric polynomials. Two obvious counter-examples are

$$x^2y^2 + x^3 + y^3 + xy = (x^2+y)(y^2+x), \quad \text{and}$$
$$xy^2 + x^2y + yz^2 + y^2z + zx^2 + z^2x + 2xyz = (x+y)(y+z)(z+x).$$

Another method that is often useful for finding symmetric but not necessarily homogeneous factors is the following:

Let a, b, c be the roots of

$$x^3 - px^2 + qx - r = 0.$$

Then

$$\sum a^2 = p^2 - 2q,$$
$$\sum a^3 = p\sum a^2 - q\sum a + 3r = p^3 - 3pq + 3r.$$

Whence,

$$a^3 + b^3 + c^3 - 3abc = p^3 - 3pq = p(p^2 - 3q) = \left(\sum a\right)\left(\sum (a^2 - ab)\right).$$

Comment on Q329

Q329. [January, 1964]. *Comment by Charles Ziegenfus, Madison College, Virginia.*

In attempting to solve Q329, I found that its solution was arrived at very quickly by normal factorization. In this case I believe the straightforward approach is somewhat simpler than the proposed solution.

$$\begin{aligned}1.\ n^4 + 2n^3 + 2n^2 + 2n + 1 \\ &= (n^4 + 2n^2 + 1) + 2n(n^2 + 1) \\ &= (n^2 + 1)^2 + 2n(n^2 + 1) \\ &= (n^2 + 1)(n^2 + 1 + 2n) \\ &= (n^2 + 1)(n + 1)^2\end{aligned}$$

$$\begin{aligned}2.\ n^4 + 2n^3 + 2n^2 + 2n + 1 \\ &= (n^4 + 4n^3 + 6n^2 + 4n + 1) - 2n(n^2 + 2n + 1) \\ &= (n + 1)^4 - 2n(n + 1)^2 \\ &= (n + 1)^2(n^2 + 2n + 1 - 2n) \\ &= (n + 1)^2(n^2 + 1).\end{aligned}$$

Comment on Q331

Q331. [March, 1964]. *Comment by J. A. H. Hunter, Toronto, Ontario, Canada.*

Referring to the published answer to Q331 in the March 1964 issue, I was disappointed to see the rather clumsy convergent solution quoted. Although the standard approach, as given in most textbooks, in practice, this is rarely preferable to the more elegant and more *manageable* recurrence solution for equations of the Pell type.

Say we have $x^2 - Ny^2 = -1$, where of course N is not a square. In general, the textbooks deal with this by one of two methods:

(1) Calculation of penultimate convergents in the continued fraction that is derived for $\sqrt{N}$.

(2) Derivation, for successive values of n, of the general solution:

$$x = [(b\sqrt{N} + a)^{2n+1} - (b\sqrt{N} - a)^{2n+1}]/2$$
$$y = [(b\sqrt{N} + a)^{2n+1} + (b\sqrt{N} - a)^{2n+1}]/2\sqrt{N},$$

where $x = a$, $y = b$ is the first solution of $x^2 - Ny^2 = -1$. Both of these methods become impracticable beyond the first two or three pairs of values, involving an inordinate amount of numerical working.

There is a very preferable method which can be used with a minimum of numerical working for all cases of $x^2 - Ny^2 = -1$ (as defined above). For reasons which I have never understood, this elegant approach is generally ignored in the textbooks, so a very brief outline may be in order here.

Assuming that $x^2 - Ny^2 = -1$, we first determine the smallest integral solution (if there be a solution) by trial or by the convergence method: say this is $x=a$, $y=b$.

Next, by trial or the convergence method, we determine the smallest *nonzero* solution for $x^2-Ny^2=+1$: say this entails $x=e$.

Then, all solution of $x^2-Ny^2=-1$ are provided by:

$$x = \pm (am \pm Nbn),$$
$$y = \pm (bm \pm an),$$

where $m^2-Nn^2=1$.

The first two pairs of values in $m^2-Nn^2=+1$ have been determined already as $(m, n)=(1, 0)$ and (e, f), say. All further successive pairs of solutions will be given by use of the recurrence relation which applies to both m and n:

$$U_r = 2eU_{r-1} - U_{r-2}, \quad \text{subject to } r > 2.$$

From these (m, n) values the corresponding (x, y) values can be determined.

However, it is not necessary to continue the derivation of successive (x, y) values by this method beyond the *second smallest* pair. For, having determined the two smallest pairs for (x, y), we can immediately derive the successive further pairs by using that same recurrence relation with respect to both x and y (without recourse to $m^2-Nn^2=+1$).

Q331 provides a simple example of this procedure. We had $X^2-2Y^2=-1$, where $X=2x+1$, $Y=y+1$. The smallest solution of this is $X=1$, $Y=1$: i.e., $a=1$, $b=1$. Then the smallest nonzero solution of $X^2-2Y^2=+1$, is $X=3$, $Y=2$. So, the first two pairs of values in $m^2-2n^2=+1$, are $(m, n)=(1, 0)$ and $(3, 2)$.

Correspondingly the two smallest pairs in $X^2-2Y^2=-1$, are:

$$X = 1, \quad Y = 1, \quad \text{and} \quad X = \pm (3 \pm 4), \quad Y = \pm (3 \pm 2).$$

Also, we have ascertained that in this case, $2e=6$ (from the smallest nonzero solution of $X^2-2Y^2=+1$). Hence, we can now tabulate successive solutions for $X^2-2Y^2=-1$ without undue numerical working as:

$$X = 2x + 1 = 1,\ 7,\ 41, \text{ etc.}$$
$$Y = \quad y + 1 = 1,\ 5,\ 29, \text{ etc.}$$

Both subject to:

$$U_r = 6U_{r-1} - U_{r-2}, \qquad \text{for } r > 2.$$

In fact, it will be found that a similar recurrence relation will give the corresponding (x, y) values without actually noting the successive values of X and Y:

$$x = 3,\ 20,\ 119, \text{ etc.} \qquad x_r = 6x_{r-1} - x_{r-2} + 2, \qquad \text{for } r > 2.$$
$$y = 4,\ 28,\ 168, \text{ etc.} \qquad y_r = 6y_{r-1} - y_{r-2} + 4, \qquad \text{for } r > 2.$$

The practical application of this concept is explained in rather more detail in *Mathematical Diversions* by J. A. H. Hunter and Joseph S. Madachy, in the chapter, *Diophantos and All That.*

Comment on F20

F20. [January, 1964]. *Comment by William Squire, West Virginia University.*

Trigg's result can be generalized to give:

$$\left[a+\frac{a}{a^n-1}\right]^{1/n}=a\left[\frac{a}{a^n-1}\right]^{1/n}.$$

Thus

$$\left(2\,\frac{2}{3}\right)^{1/2}=2\left(\frac{2}{3}\right)^{1/2},\qquad \left(3\,\frac{3}{8}\right)^{1/2}=3\left(\frac{3}{8}\right)^{1/2}$$

$$\left(2\,\frac{2}{7}\right)^{1/3}=2\left(\frac{2}{7}\right)^{1/3},\qquad \left(3\,\frac{3}{26}\right)^{1/3}=3\left(\frac{3}{26}\right)^{1/3}$$

$$\left(2\,\frac{2}{15}\right)^{1/4}=2\left(\frac{2}{15}\right)^{1/4},\qquad \left(3\,\frac{3}{80}\right)^{1/4}=3\left(\frac{3}{80}\right)^{1/4},\ \text{etc.}$$

It can be shown that the formula represents the only cases provided the fraction is reduced, i.e.,

$$2\,\frac{4}{6}=2\,\frac{2}{3},\ \text{etc.}$$

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q340. Determine $\{a_r\}$ such that

$$[a_0+a_1+a_2+\cdots]^x=a_0+a_1x+a_2x^2+\cdots.$$

[*Submitted by M. S. Klamkin.*]

Q341. Find the limit of the fraction as n approaches infinity:

$$\frac{\phi(1)+\phi(2)+\phi(3)+\cdots+\phi(n)}{1+2+3+\cdots+n}$$

where $\phi(n)$ is Euler's totient.

[*Submitted by Huseyin Demir.*]

Q342. Which is larger $\sqrt[9]{9!}$ or $\sqrt[10]{10!}$?

[*Submitted by D. L. Silverman.*]

Q343. Identify the angle θ satisfying

$$\frac{\sin(1\cdot\theta+15^\circ)}{\sqrt{1}}=\frac{\sin(2\cdot\theta+15^\circ)}{\sqrt{2}}=\frac{\sin(3\cdot\theta+15^\circ)}{\sqrt{3}}.$$

[*Submitted by Huseyin Demir.*]

(Answers on page 252)

ANSWERS

A340.

$$[a_0+a_1+a_2+\cdots]^x = e^{x\log S} = 1+\frac{x\log S}{1!}+\frac{x^2\log^2 S}{2!}+\cdots,$$

where $S=\sum a_r$. Consequently we have

$$a_0 = 1, \qquad a_2 = \frac{a_1^2}{2!}, \cdots, a_r = \frac{a_1^r}{r!}$$

and $S=e^{a_1}$.

A341. $\phi(k)$ denotes the number of integers smaller than k and prime to it. Hence,

$$\sum_1^n \phi(k)$$

is the total number of relatively prime pairs among the first n integers, the total number of pairs being $\binom{n}{2}$. The limit of the given fraction being

$$\sum_1^n \phi(k) \Big/ \binom{n}{2}$$

it will be the probability that any two integers taken at random be relatively prime. This probability is known to have the value $6/\pi^2$. Hence, the limit of the given fraction is $6/\pi^2$.

A342. Since $\sqrt[n]{n!}$ is the geometric mean of 1, 2, 3, $\cdots$, n, it is obviously an increasing function of n, so

$$\sqrt[9]{9!} < \sqrt[10]{10!}$$

A343. The angle θ is evidently 15°.

(Quickies on page 286)

CORRECTION

In the paper "A Theorem on Integer Quotients of Products of Factorials," this MAGAZINE, 36 (1963) 98, insert between $[l_n]=n$ and Theorem 1 the following sentence:

Assume that if

$$[l_n] = [l_a] + [l_b] + [l_c] + \cdots + [l_k],$$

then

$$\left[\frac{l_n l_m}{l_r}\right] \geqq \left[\frac{l_a l_m}{l_r}\right] + \left[\frac{l_b l_m}{l_r}\right] + \cdots + \left[\frac{l_k l_m}{l_r}\right].$$

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